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Fuzzy Model of 16PSK and 16QAM Modulation

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Abstract—**In the paper, a concept of Additive Fuzzy Noise (AFN) channel is introduced. The theoretical equations are derived for Bit Error Rate (BER) and Symbol Error Rate (SER) with some digital modulation scheme in the AFN channel. Following modulations are considered: Phase Shift Keying (16PSK), Quadrature Amplitude Modulation (16QAM). The fuzzy approach to these modulations is presented. The BER and SER values are calculated using possibility theory. The results obtained by fuzzy noise model are compared with conventional approach, where probability models of the noise are used.**

Keywords—**Fuzzy modulation, BER, possibility, probability, PSK, QAM, noise, signal.**

I. INTRODUCTION

A DISCRETE telecommunication channel consists of a mapping device, transmit filter, propagation path, demodulator, and decision device. Conventionally, is described by signal theory or information theory using probability methods. In this paper a definition of the Additive Fuzzy Noise (AFN) channel is suggested. The fuzzy channel consists of similar devices as conventional, thus its structure is similar to a conventional channel. However, fuzzy description is applied. It differs substantially from probability description. A digital modulation involves choosing a particular signal form a finite set of possible signals. During a transmission of the signal by a channel additional effects occur as interference between signals from other sources, distortions, noise, phase jitter, etc. Many theoretical and practical solutions were proposed in order to obtain correct recovering of transmitted messages. The description of such solutions can be found in basic books from telecommunication area, see for example [1]–[3]. Some of the errors mentioned above can be diminish by appropriate construction of the transmitter and receiver. However, such effects as noise cannot be eliminated because it weakly depends on equipment framework. The noise can be diminished by choice of frequency band, but total elimination is not possible. Influence of the noise is different and depends on type of signal modulation. As a model of noise the Additive White Gaussian Noise (AWGN) was assumed most frequently [2] or sometimes Reighlay noise (fading).

For seven years, the author has been trying to establish new theory of fuzzy signals. Basing concepts of signal theory as fuzzy signal spectrum, based on proposed Fuzzy Fourier Transform (FuzFT) [4], [5], fuzzy correlation and fuzzy convolution [6], fuzzy time invariant systems [7], fuzzy digital filters [8], [9] were suggested.

In this paper the author describes an approach to concept of fuzzy noise and possibilistic analysis of the error rates for digital modulations used currently in the telecommunication equipment. In his previous paper, sent to publication [10], the author described the following modulations: Binary Phase Shift Keying (BPSK), Binary Frequency Shift Keying (BFSK), Pulse Amplitude Modulation (4PAM), Quadrature Phase Shift Keying (QPSK) and Quadrature Amplitude Modulation (QAM). The Bit Error Rate (BER) and Symbol Error Rate (SER) values were calculated using possibility theory. The results obtained by conventional approach, where probability model of the noise was used, were compared with the results obtained by simulation of appropriate fuzzy noise.

Here, two models are proposed. First model is fuzzy. It was assumed that the symbols sent at the input of the channel are crisp functions of time. The noise and the output of the channel are fuzzy functions of time. Analysis of the following types of modulation are presented: 16PSK and 16QAM. The possibility of symbol error rates for these types of modulations are calculated and discussed. The results are compared with that obtained by probabilistic methods for AWGN. Second model is fuzzy-random. Some recent results for fuzzy-random theory can be found in [11]. Here, the noise and interference are supposed as random functions. The fuzzy model describes their influence on the symbol error rate and decision taken by the decision device.

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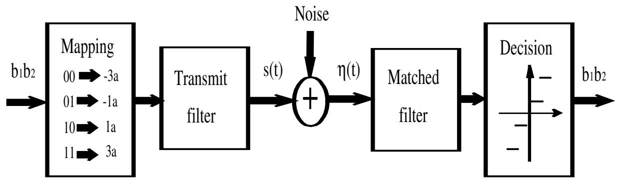

Fig. 1. Fuzzy channel.

II. FUZZY CHANNEL

The conventional approach to digital channel description is based on probability model. The series of input bytes $b_1, b_2, ...$ enters at coding device, is converted into symbols $s(t)$ depending on type of modulation, and send to the channel. The signal arriving at the input of digital receiver, i.e. at the output of the channel, consists of a sum of transmitted signal, multiplied by a constant value, and noise. Decision device ought to recover correct series of input bytes (see Fig. 1). It is supposed that symbols $s(t)$ entering into the channel are crisp functions of the time t. The fuzzy channel is linear and satisfies the equation

$$\eta(t) = hs(t) + n(t) \tag{1}$$

where $\eta(t)$ is output of a channel, h is a scaling factor, $s(t)$ is a symbol and $n(t)$ is a noise. The equation describing the conventional crisp channel is similar. Nevertheless, the interpretation will be quite different. It is possible to consider all terms in equation (1) as fuzzy functions or fuzzy numbers, but such interpretation is very complicated and not necessary. It is sufficient to consider $n(t)$ as fuzzy function, which will be called fuzzy noise, and output $\eta(t)$ as fuzzy function. The channel scaling factor h, the modulated signal $x(t)$, and symbol $s(t)$ can be crisp. A decision taken at any discrete moment kT, k=0,1,2,... at the end of transmission chain is discrete. The noise has influence on this decision, but because of discretization an assumption can be introduced that only mean value of the noise during period T is important, not all instant values $n(t)$. Therefore, in following part of the paper noise n is treated as fuzzy number with membership function $\mu_n(v)$. Thus, the output of the channel be fuzzy complex number and will be denoted by η. A channel satisfying such assumptions with additive fuzzy noise will be called the additive fuzzy discrete channel (AFDC).

III. PSK Modulation

A. Probabailistic Approach

When baseband modulation is performed on an information bit stream $b_1, b_2, ,$ the number of bits encoded per symbol is constant $log_2 M$, with M representing the size of the symbol set. In this case the complex baseband PSK modulation signal is

$$s_k(t) = \sqrt{E_s} exp[-j2\pi k/M + \Theta_0] \tag{2}$$

where E_s is the energy per symbol, Θ_0 is a constant phase offset, and $k = 0, ..., M-1$. The constellation diagram for 16-PSK (M=16) is shown in Fig. 2. Conventional approach to Bit Error Rate (BER) applies probability theory. Typically, the noise in (1) is assumed as Additive White Gaussian Noise (AWGN) with constant power bilateral spectral density $N_0/2$. Assuming 2-dimensional Gaussian probability density function $p(x, y)$ for the noise, the expression for BER, when M-PSK modulation is used, can be calculated. Unfortunately, general expression is complicated and can be found in [1]. For big values E_s/N_0 and $M \geq 8$ simplified form can be obtained

$$Prob_{MPSK} \approx erfc[\sqrt{E_s/N_0}\sin(\pi/M)] \tag{3}$$

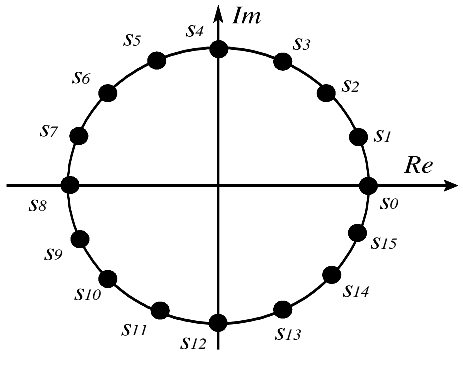

Fig. 2. Constellation for 16PSK modulation.

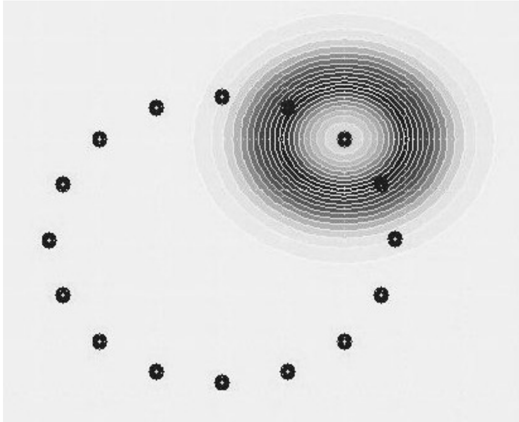

Fig. 3. Constellation for 16PSK and *pdf* for symbol s_2.

With the purpose of easy calculation, the following procedure is applied. The noise affects the real and imaginary part of signal $s(t)$. Let symbol s_k be transmitting. Assuming the conditional probability density function $p_n(x, y|s_k)$ for the noise (see Fig. 3)

$$p_n(x, y|s_k) = \frac{1}{\sqrt{\pi N_0}} \cdot \exp\left(-\frac{[x - Re(s_k)]^2 + [y - Im(s_k)]^2}{N_0}\right) \tag{4}$$

the BER value is numerically calculated using (3). The probability of error, where s_2 is received as s_1, is approximated by

$$Prob_{s2\to s1} \approx \frac{1}{2} erfc[\sqrt{E_s/N_0}\sin(\pi/M)] \tag{5}$$

and where s_2 is received as s_0

$$Prob_{s2\to s0} \approx \frac{1}{2} erfc[\sqrt{E_s/N_0}\sin(2\pi/M)] \tag{6}$$

For 16-PSK it obtains $Prob_{s2\to s1} \approx 0.1915$ and $Prob_{s2\to s0} \approx 0.0065$.

B. Possibilistic Approach

Consider now fuzzy description. It must be noted that there is great difference between probability and possibility concepts. A probability density function $pdf(x)$ fulfils the requirements:

$$pdf(x) \geq 0 \tag{7}$$

$$\int_{-\infty}^{\infty} pdf(x)dx = 1 \tag{8}$$

$$max[pdf(x)]] \neq 1 \qquad (generally) \tag{9}$$

whereas for membership function $\mu(x)$ the requirements are:

$$\mu(x) \geq 0 \tag{10}$$

$$\int_{-\infty}^{\infty} \mu(x)dx \neq 1 \qquad (generally) \tag{11}$$

$$max[\mu(x)]] = 1 \tag{12}$$

In fuzzy mathematics Zadeh [12] introduced a concept of possibility. The possibility of an event "x is A", where x is crisp value and A is a fuzzy set with membership function $\mu_A(x)$ is defined as numerically equal to $\mu_A(x)$

$$Poss\{x \; is \; A\} = \mu_A(x) \tag{13}$$

Moreover,

$$Poss\{\emptyset\} = 0 \quad Poss\{X\} = 1 \tag{14}$$

where $\emptyset$ is empty set and X is universum. For any disjoint fuzzy sets A, B

$$Poss\{A \cup B\} = max[Poss\{A\}, Poss\{B\}] \tag{15}$$

$$Poss\{A \cap B\} = min[Poss\{A\}, Poss\{B\}] \tag{16}$$

Generally, maximum and minimum can be replaced by more general operations, triangular norms ($t - norm$, $s - norm$). Thus, the formula are quite different from probabilistic.

Let us try to compare fuzzy approach with probabilistic one. There are three main methods for probability-possibility transformation:

- conserving the same entropy
- transforming confidence interval into α-cuts of fuzzy set
- conserving similar shapes of $pdf(x)$ and membership function $\mu(x)$.

First method is applied to conserve similar amount of information. Second, transforms any confidence interval at level α into appropriate α-cut of fuzzy set. Third, is very simple, conserves second moment in one point, but not exact.
A graphical comparison of second and third methods is shown in Fig. 4 where normal distribution is transformed into membership function.

Sudkamp [13] shows that do not exist any transformation conserving properties of higher order, as moments. However, it is possible to transform the pdf $p_n(x, y|s_k)$ into membership $\mu_n(x, y|s_k)$ conserving the Gaussian shape, with the same central value, but with $max(\mu_n(x, y|s_k) = 1$, and change the variance of the noise $var = \sqrt{N_0/2}$ in such a way to obtain identical $Prob_{s_2 \to s_1} = Poss_{s_2 \to s_1}$. Let E_s=10, N_0=1, then $var = 0.707$. After this transformation

$$\mu_n(x, y|s_k) = \exp\left(-\frac{[x - Re(s_k)]^2 + [y - Im(s_k)]^2}{2var_1}\right) \tag{17}$$

where Re and Im are real and imaginary parts of symbol, $var_1 = 2.303 * var = 1.628$. Fig. 3 shows $\mu(x, y|s2)$. The possibility of erroneous transmission $s_2 \to s_1$ occurs when output value of the channel lies in the area limited by radiant rays with angles $\pi/8 \pm \pi/16$ (see Fig. 5). Let denote this area by A. The possibility $Posss_{2 \to s_1}$ is equal to maximal value of the membership $\mu(x, y|s_2)$ in A. The maximum is attained in the point lying in half a way between s_2 and s_1. Thus, really

$$Poss_{s2 \to s1} = exp\Big\{ - \Big([Re(\frac{s_2 - s_1}{2})]^2 + [Im(\frac{s_2 - s_1}{2})]^2\Big)/2var_1\Big\} = 0.1915 \tag{18}$$

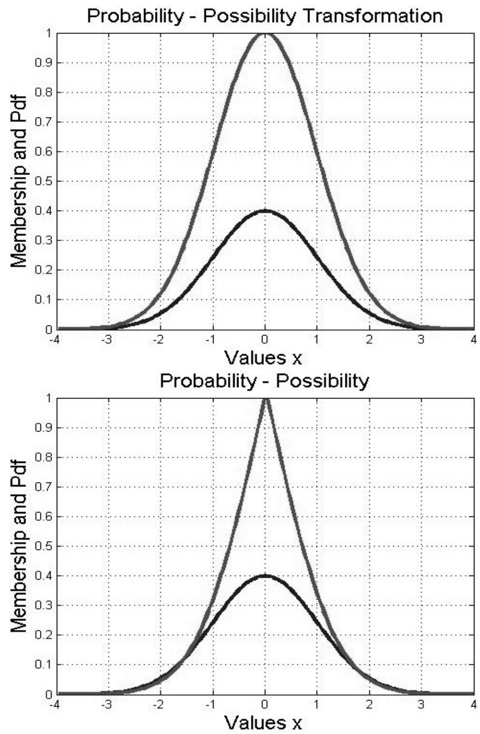

Fig. 4. Two types of probability-possibility transformation, similar shape type (up) and confidence type (down).

Using the same variance $var_1 = 1.628$ as before it obtains for the event $s_2 \to s_0$

$$Poss_{s2 \to s0} = exp\Big\{ - \Big([Re(\frac{s_2 - s_0}{2})]^2 + [Im(\frac{s_2 - s_0}{2})]^2\Big)/2var_1\Big\} = 0.0044 \tag{19}$$

The same results are right for any transition $s_k \to s_{k\pm1}$ and $s_k \to s_{k\pm2}$.

The value obtained for $Poss_{s_2 \to s_1}$ is equal to $Prob_{s_2 \to s_1}$ on account of our assumption, but $Poss_{s_2 \to s_0} = 0.0044$ is quite different from $Prob_{s_2 \to s_0} = 0.0065$. The discrepancy arises by reason of different operations used for probability (integration) and possibility (maximization) Therefore, using Gaussian membership it is not possible to obtain similar results. The membership function must have different shape.

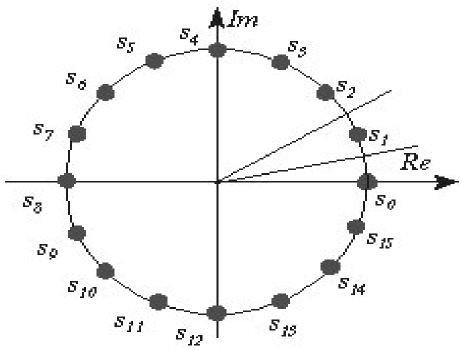

Fig. 5. The method for calculation of BER.

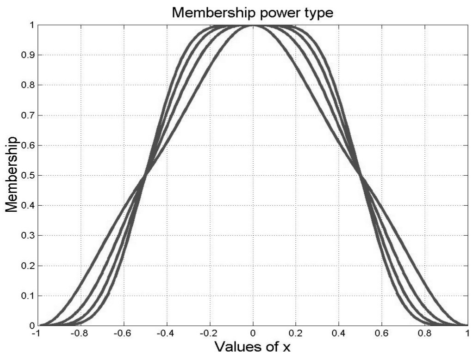

Fig. 6. The power-type membership family of fuzzy sets.

It may be considered as very inconvenient. However, Gaussian pdf is only rough approximation with only one parameter – variance, because the mean value must be s_2. Moreover, if some experimental results are known, the possibilistic approach is more reasonable and membership function can be easily built with accordance to data. Also, infinite limits in Gaussian pdf are not reliable.

If second method, transformation of the confidence interval into α-cuts, is used then first confidence level $\alpha(x)$ for symmetric pdf is calculated for any interval $[0, x]$

$$\alpha(x) = \int_0^x pdf(u)du \tag{20}$$

Next, the membership is found as

$$\mu(x) = [1 - 2\alpha(x)] \tag{21}$$

Applying this method identical results for BER were obtained as for probabilistic approach $Poss_{s_2 \to s_1} = Prob_{s_2 \to s_1} = 0.1961$ and $Poss_{s_2 \to s_0} = Prob_{s_2 \to s_0} = 0.0074$. Small discrepancy between both methods (0.1915 and 0.1961) is caused by different approximation methods, which were used for simplifying calculations. Some discussion about approximation for BER calculation can be found in [14].

For the membership function many different shapes can be applied: triangle, trapeze, bell function: The membership function can be built directly, without any transformation or approximation method, with accordance to measured data. It is easier than probabilistic approach, where requirement (8) must be satisfied. If analytic form of membership is necessary, the good power type family of functions can be used. It is defined by

$$\mu(x) = \begin{cases} 1/(1 + |2x|^n) & for \ \ |x| \le 1/2 \\ 1 - 1/\big(1 + [2(1 - |x|)]^n\big) & for \ \ 1/2 < x < 1 \\ 0 & for \ \ |x| \ge 1 \end{cases}$$

where parameter $n \ge 1$. Of course, the interval $x \in [-1, 1]$ can be enlarged. The family is shown in Fig. 6.

IV. QAM Modulation

A. Probabailistic Approach

The QAM constellation is not unique. Circular or rectangular schema is used. Two examples are shown in Fig. 7. In the paper rectangular 16-QAM constellation will be consider. In this case the calculation is somewhat more complicated then for PSK modulation. The symbols s_k have complex values

$$\sqrt{E_s/N_0}(\pm 1 \pm j), \quad \sqrt{E_s/N_0}(\pm 3 \pm j3)$$

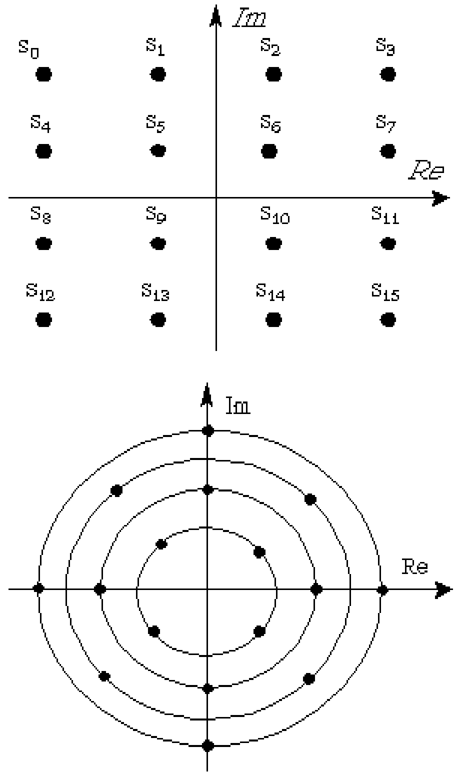

Fig. 7. Rectangular and circular constellation for 16-QAM modulation.

The probability or possibility of error depends on which symbol is transmitted. For example s_3 has smaller probability of error than s_6 because it is surrounded only by 3 symbols not by 8 other symbols as s_6. First consider probabilistic approach. Let the symbol s_6 be transmit. Fig. 8 shows such case. Following the analysis presented in [14] the correct transmission occurs when output signal fulfill conditions

$$0 < Re(\eta) < 2\sqrt{E_s/10}) \quad 0 < Im(\eta) < 2\sqrt{E_s/10}) \tag{22}$$

Let the conditional pdf for the real part be

$$Prob(x|s_6) = \frac{1}{\sqrt{\pi N_0}} exp\big[-\frac{(x - \sqrt{E_s/10})^2}{N_0}\big] \tag{23}$$

and similarly for imaginary part. Assuming statistical independence of the real and imaginary parts, it follows that for the probability of correct transmission it obtains

$$Prob(c|s_6) = \Big[1 - erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big]^2 \tag{24}$$

Thus, the probability of an error

$$Prob(e|s_6) = 1 - \Big[1 - erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big]^2 \approx 2erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) \tag{25}$$

Now let the symbol s_3 be transmitting. Correct transmission occurs when

$$Re(\eta) > 2\sqrt{E_s/10}) \quad Im(\eta) > 2\sqrt{E_s/10}) \tag{26}$$

Therefore, probability of the error

$$Prob(e|s_3) = 1 - \Big[1 - \frac{1}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big]^2 \approx erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) \tag{27}$$

Similar analysis for symbol s_7 gives

$$Prob(e|s_7) = 1 - \Big[1 - \frac{1}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big] \cdot \Big[1 - erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big] \approx \frac{3}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) \tag{28}$$

Total probability of an error is weighted mean and equals

$$Prob_{16QAM} \approx \frac{4}{16}2erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) + \frac{4}{16}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) + \frac{8}{16}\frac{3}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) = \frac{3}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big) \tag{29}$$

Moreover, the erroneous transition $s_6 \to s_7$ occurs with probability

$$Prob(s_6 \to s_7) = \int_{2\sqrt{E_s/10}}^{\infty} Prob(x|s_6)dx \cdot \int_{0}^{2\sqrt{E_s/10}} Prob(y|s_6)dy = \frac{1}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big[1 - erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big] \tag{30}$$

and transition $s_6 \to s_3$ with probability

$$Prob(s_6 \to s_3) = \int_{2\sqrt{E_s/10}}^{\infty} Prob(x|s_6)dx \cdot \int_{2\sqrt{E_s/10}}^{\infty} Prob(y|s_6)dy = \Big[\frac{1}{2}erfc\Big(\sqrt{\frac{E_s}{10N_0}}\Big)\Big]^2 \tag{31}$$

Let the energy of symbol be $E_s = 10$ and power spectral density of the noise $N_0 = 1$, For correct transmission of s_6

$$Prob(c|s_6) = [1 - erfc(1)]^2 = 0.7101 \tag{32}$$

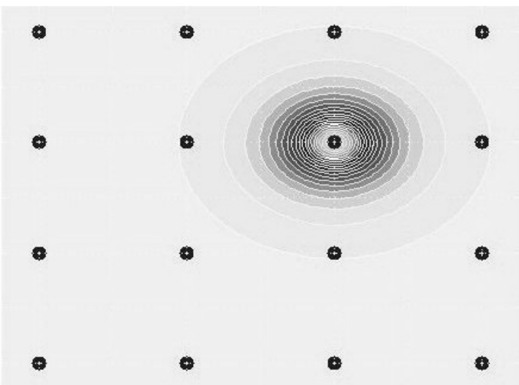

Fig. 8. The pdf for 16-QAM modulation when symbol s_6 was transmit.

Thus, error transmission occurs with probability $Prob(e|s_6) = 0.2899$. The probability of the error when symbol s_3 is transmit is expressed as

$$Prob(e|s_3) = 1 - [1 - erfc(1)/2]^2 = 0.1511 \tag{33}$$

and for total probability of an error for 16QAM transmission is weighted mean (see [14]) and is equal

$$Prob_{16QAM} \approx (3/2)erfc(1) = 0.2359 \tag{34}$$

For probability of transition $s_6 \to s_7$ it obtains

$$Prob(s_6 \to s_7) = erfc(1)[1 - erfc(1)]/2 = 0.0663 \tag{35}$$

For probability of transition $s_6 \to s_3$ it obtains

$$Prob(s_6 \to s_3) = [erfc(1)]^2/4 = 0.0062 \tag{36}$$

B. Possibilistic Approach

Consider now fuzzy description. Let membership function be similar as in equation (17), but now variation will be different, when similar shape method is applied for transformation probability-possibility. The possibility of correct transmission occurs when (22) is satisfied. Therefore, possibility of erroneous transmission $s_6 \to s_7$

$$Poss_{s_6 \to s_7} = exp\Big\{ - \frac{\big[Re(s_6 - s_7)\big]^2}{8var_1}\Big\} \tag{37}$$

and similarly the possibility of the error $s_6 \to s_2$ has value

$$Poss_{s_6 \to s_2} = exp\Big\{ - \frac{\big[Im(s_6 - s_2)\big]^2}{8var_1}\Big\} \tag{38}$$

The possibility of the error $s_6 \to s_3$

$$Poss_{s_6 \to s_3} = exp\Big\{ - \Big(\Big[Re(\frac{s_6 - s_3}{2})\Big]^2 + \Big[Im(\frac{s_6 - s_3}{2})\Big]^2\Big)/2var_1\Big\} \tag{39}$$

Applying transformation probability-possibility using similar shape method, where arbitrary condition $Prob(s_6 \to s_7) = Poss(s_6 \to s_7)$ is supposed, it follows that variation of membership function must be equal $var_1 = 0.3685 * var$. In such situation a discrepancy of other results arises because
$Poss_{s_6 \to s_7} = Prob(s_6 \to s_7) = 0.0663$
$Poss_{s_6 \to s_3} = 0.0044 \neq Prob(s_6 \to s_3) = 0.0062$
Thus, agreement is satisfied only in some points.

Now, let confidence method be applied. The membership function is not normal and shape is similar as presented in Fig. 4. Applying (20) and (21) it obtains $Posss_6 \to s_7 = 0.0663$ and $Poss_{s_6 \to s_3} = 0.0062$. Both results agreed with probabilistic.

V. Conclusion

In the paper the differences between probabilistic and possibilistic approach was shown. A discrepancy of the results is presented where similar shape method of probability-possibility transformation is applied. It is caused by another interpretation of probability density function and membership function. Simple transformation of the probability density function into membership function, with conservation of similar shape of function, for example normal distribution into normal membership changing only the amplitude and variance, can preserve the similar value of error in one point but it will cause a discrepancy in other points. Thus, the membership function must have another shape. For confidence interval method, proposed by Dubois *et al.*, the results of probabilistic and possibilistic are agreed. If after gathering of measured data the membership function will be built correctly as possibility of errors then discrepancy can be avoided. Any approximation by analytic special function as gaussian is not necessary. Thus, possibilistic approach is easier.

In more complicated cases, when conjunction of independent random events occurs, in probability theory total probability is calculated as product of probabilities. In the case of conjunction of fuzzy sets t-norms operations are applied, commonly minimum. In such case the result will not agreed with probabilistic. The product operation must be applied as t-norm to avoid the discrepancy of results.

It must be noted that apart from presented approach there exists another probability-possibility transformation proposed by De Luca and Termini [15]. It is based on membership function. In future work the author will consider this concept.

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