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Investigation of the Stability and Convergence of Difference Schemes for the Three-dimensional Equations of the Atmospheric Boundary Layer

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***Abstract*—In this article we construct a finite-difference scheme for the three-dimensional equations of the atmospheric boundary layer. The solvability of the mathematical model is proved and quality properties of the solutions are studied. A priori estimates are derived for the solution of the differential equations. The mathematical questions of the difference schemes for the equations of the atmospheric boundary layer are studied. Nonlinear terms are approximated such that the integral term of the identity vanishes when it is scalar multiplied. This property of the difference scheme is formulated as a lemma. Main a priori estimates for the solution of the difference problem are derived. Approximation properties are investigated and the theorem of convergence of the difference solution to the solution of the differential problem is proved.**

***Keywords*— atmospheric boundary layer equations, difference scheme, approximation error, stability, convergence algorithm, numerical solution.**

I. Introduction

MATHEMATICAL models of computational fluid dynamics serves as the basis for the study of various natural phenomena, technological processes and environmental problems. In this regard, the development and study of efficient and stable numerical algorithms for solving the system of equations of the atmospheric boundary layer and their practical implementation is relevant. There are various methods for the numerical solution of differential equations, new techniques have been developing, the work on their improvement has been continuously performed, and reassessing the methods is carried out. Basic methods for solving grid equations are systematized and described in detail in [1]. When solving the Navier–Stokes equations, explicit schemes are inefficient due to hard restrictions on the ratio of temporal and spatial steps of the computational grid, especially on finding stationary establishing solutions. Therefore, the most frequently used implicit difference scheme is unconditionally stable or has weaker constraints on the stability. An overview of the most commonly used numerical algorithms is presented e.g. in papers [2]-[10].

In [11] a new symmetric method of approximation of the non-stationary Navier-Stokes equations of the Cauchy-Kovalevskaya type is proposed. The properties of the modified problem are studied. The convergence of the solution of modified problem to the solution of the original problem is proved on an infinite time interval when $\varepsilon \to 0$. In [12] the convergence of a finite-difference scheme, approximating the primitive equations with the second order in the spatial variables, to the solution of the differential problem is proved under the natural assumption of smoothness of the solution of the original problem. Paper [13] studies difference schemes by time which accuracy order can be arbitrarily high. Difference schemes by time for solving the Navier-Stokes equations are presented. The impact of the scheme order on the calculations accuracy is examined. In [14]-[16] numerical algorithms for solving the Navier-Stokes equations using the finite element method are proposed. The analysis of stability and convergence of the proposed methods is conducted.

In [17], stable and convergent difference schemes for the boundary layer equations of the atmosphere, transport and transformation of impurities of harmful substances were constructed. A package of applied programs for the numerical simulation of atmospheric air pollution taking into account photochemical transformations and visualizations of the corresponding scenarios was developed. The problem of impurities' distribution from point sources was considered. The results of numerical modeling of the harmful impurities' propagation and transformation on the mesometeorological processes were presented on the example of Ust-Kamenogorsk.

II. Statement of the Problem

Consider the three-dimensional equations of the atmospheric boundary layer in a domain $\Omega = \{0 < x_i < l_i, i = 1, 2, 3\}$ with a border S :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + \upsilon\frac{\partial u}{\partial x_2} + \omega\frac{\partial u}{\partial x_3} + \frac{\partial p}{\partial x_1} = \frac{1}{De}\upsilon + \frac{1}{\mathrm{Re}_T}\left(\frac{\partial}{\partial x_1}\left(a_{x_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(a_{x_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(a_{x_3}\frac{\partial u}{\partial x_3}\right)\right) + f_1(\vec{x},t) \qquad (1)$$

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$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x_1}+\upsilon\frac{\partial \upsilon}{\partial x_2}+\omega\frac{\partial \upsilon}{\partial x_3}+\frac{\partial p}{\partial x_2}=-\frac{1}{De}u+ \tag{2}$$
$$+\frac{1}{\mathrm{Re}_T}\left(\frac{\partial}{\partial x_1}\left(a_{x_1}\frac{\partial \upsilon}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a_{x_2}\frac{\partial \upsilon}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(a_{x_3}\frac{\partial \upsilon}{\partial x_3}\right)\right)+f_2(\vec{x},t),$$

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x_1}+\upsilon\frac{\partial \omega}{\partial x_2}+\omega\frac{\partial \omega}{\partial x_3}+\frac{\partial p}{\partial x_3}=\bar{\lambda}+ \tag{3}$$
$$+\frac{1}{\mathrm{Re}_T}\left(\frac{\partial}{\partial x_1}\left(a_{x_1}\frac{\partial \omega}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(a_{x_2}\frac{\partial \omega}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(a_{x_2}\frac{\partial \omega}{\partial x_3}\right)\right)+f_3(\vec{x},t),$$

$$div\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial \omega}{\partial z}=0 \tag{4}$$

where t is time, x_1, x_2, x_3 are Cartesian coordinates, $\vec{V}$ is the wind velocity vector with components u,υ,ω, p is pressure, De is a dimensionless characteristic describing deviation of the wind from the geostrophic value, Re_T is a dimensionless number of turbulent exchange, $\bar{\lambda}$ is a dimensionless parameter of convection, a_{x_1}, a_{x_2} are horizontal coefficients of atmospheric turbulence for the amount of movement, a_{x_3} is a vertical coefficient of atmospheric turbulent exchange for the amount of movement.

The system of equations (1)-(4) is complemented by the following initial and boundary conditions:

$$\vec{V}(x,0)=\vec{V}^0(x),\quad x\in\Omega;\ \vec{V}(x,t)=0,\ x\in S. \tag{5}$$

In Ω the function $\vec{V}^{\,0}(x)$ is set as follows:

$$div\vec{V}^0(x)=0.$$

For the numerical solution of the equations of the atmospheric boundary layer (1)-(4), a mesh with distributed velocities is used. In Ω, we introduce the meshes Ω_H, $\Omega_H=\Omega_h\cup\Omega_x\cup\Omega_y\cup\Omega_z$, where

$$\begin{aligned}
&\Omega_h=\{(x_{1i},x_{2j},x_{3k}),\ x_{1i}=ih_1,\ x_{2j}=jh_2,\ x_{3k}=kh_3\},\\
&(i=0,1,\ldots,N_1;\ j=0,1,\ldots,N_2;\ k=0,1,\ldots,N_3,h_1=l_1/N_1,h_2=l_2/N_2,h_3=l_3/N_3)\\
&\Omega_x=\{(x_{1i+1/2},x_{2j},x_{3k}),\ x_{1i+1/2}=(i+1/2)h_1,\ x_{2j}=jh_2,x_{3k}=kh_3,\}\\
&(i=0,1,\ldots,N_1-1;\ j=0,1,\ldots,N_2;k=0,1,\ldots,N_3,h_1=l_1/N_1,h_2=l_2/N_2,h_3=l_3/N_3)\\
&\Omega_y=\{(x_{1i},x_{2j+1/2},x_{3k}),\ x_{1i}=ih_1,\ x_{2j+1/2}=(j+1/2)h_2,x_{3k}=kh_3\},\\
&(i=0,1,\ldots,N_1;\ j=0,1,\ldots,N_2-1;k=0,1,\ldots,N_3,h_1=l_1/N_1,h_2=l_2/N_2,h_3=l_3/N_3)\\
&\Omega_z=\{(x_{1i},x_{2j},x_{3k+1/2}),\ x_{1i}=ih_1,\ x_{2j}=jh_2,x_{3k+1/2}=(k+1/2)h_3\},\\
&(i=0,1,\ldots,N_1;\ j=0,1,\ldots,N_2;k=0,1,\ldots,N_3-1,h_1=l_1/N_1,h_2=l_2/N_2,h_3=l_3/N_3).
\end{aligned} \tag{6}$$

Thus, the following difference scheme is constructed:

$$\frac{u^{n+1}_{i+1/2,j,k}-u^n_{i+1/2,j,k}}{\tau}+L^{(1)}_{1h}u^n_{i+1/2,j,k}+P^{n+1}_{x_1,i,j,k}=\frac{1}{De}\upsilon^n_{i,j+1/2,k}+ \tag{7}$$
$$+\frac{1}{\mathrm{Re}_T}\left[(a_{i,j,k}u^n_{x_1,i+1/2,j,k})_{\bar{x}_1}+(a_{i+1/2,j+1/2,k}u^n_{x_2,i+1/2,j,k})_{\bar{x}_2}+\right.$$
$$\left.+(a_{i+1/2,j,k+1/2}u^n_{x_3,i+1/2,j,k})_{\bar{x}_3}\right]+f^0_{i+1/2,j,k},$$
$$i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},k=\overline{1,N_3-1}$$

$$\frac{\upsilon^{n+1}_{i,j+1/2,k}-\upsilon^n_{i,j+1/2,k}}{\tau}+L^{(2)}_{1h}\upsilon^n_{i,j+1/2,k}+P^{n+1}_{x_2,i,j,k}=-\frac{1}{De}u^n_{i+1/2,j,k}+ \tag{8}$$
$$+\frac{1}{\mathrm{Re}_T}\left[(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,i+1/2,j+1/2,k})_{\bar{x}_1}+(a_{i,j+1,k}\upsilon^n_{x_2,i,j,k})_{\bar{x}_2}+\right.$$
$$\left.+(a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,i,j+1/2,k+1/2})_{\bar{x}_3}\right]+f^0_{i,j+1/2,k},$$
$$i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},k=\overline{1,N_3-1}$$

$$\frac{\omega^{n+1}_{i,j,k+1/2}-\omega^n_{i,j,k+1/2}}{\tau}+L^{(3)}_{1h}\omega^n_{i,j,k+1/2}+P^{n+1}_{x_3,i,j,k}=\bar{\lambda}+ \tag{9}$$
$$+\frac{1}{\mathrm{Re}_T}\left[(a_{i+1/2,j,k+1/2}\omega^n_{x_1,i+1/2,j,k+1/2})_{\bar{x}_1}+(a_{i,j+1/2,k}\omega^n_{x_2,i,j+1/2,k})_{\bar{x}_2}+\right.$$
$$\left.+(a_{i,j,k+1}\omega^n_{x_3,i,j,k})_{\bar{x}_3}\right]+f^0_{i,j,k+1/2},$$
$$i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},k=\overline{1,N_3-1}$$

The continuity equation in a difference form is written as follows:

$$div_h\vec{V}^{n+1}=u^{n+1}_{\bar{x}_1,i+1/2jk}+\upsilon^{n+1}_{\bar{x}_2,ij+1/2k}+\omega^{n+1}{}_{\bar{x}_3,ijk+1/2}=0. \tag{10}$$

The following initial and boundary conditions are satisfied:

$$\begin{aligned}
&u^0_{i+1/2,j,k}=V^0(x_{1i}+0{,}5h_1,x_{2j},x_{3k}),\upsilon^0_{i,j+1/2,k}=\\
&=V^0(x_{1i},x_{2,j}+0{,}5h_2,x_{3k}),\omega^0_{i,j,k+1/2}=V^0(x_{1i},x_{2j},x_{3k}+0{,}5h_3),\\
&\upsilon^{n+1}_{0,j+1/2,k}=\upsilon^{n+1}_{N_1,j+1/2,k}=u^{n+1}_{1/2,j,k}=u^{n+1}_{N_1-1/2,j,k}=\\
&=\omega^{n+1}_{0,j,k+1/2}=\omega^{n+1}_{N_1,j,k+1/2}=0,\ j=\overline{0,N_2-1},k=\overline{0,N_3-1},\\
&\upsilon^{n+1}_{i,1/2,k}=\upsilon^{n+1}_{N_1,j-1/2,k}=u^{n+1}_{i+1/2,0,k}=u^{n+1}_{i+1/2,N_2,k}=\\
&=\omega^{n+1}_{i,0,k+1/2}=\omega^{n+1}_{i,N_2,k+1/2}=0,i=\overline{0,N_1-1},k=\overline{0,N_3-1},\\
&\upsilon^{n+1}_{i,j+1/2,0}=\upsilon^{n+1}_{i,j+1/2,N_3}=u^{n+1}_{i+1/2,j,0}=u^{n+1}_{i+1/2,j,N_3}=\\
&=\omega^{n+1}_{i,j,1/2}=\omega^{n+1}_{i,j,N_3-1/2}=0,\ i=\overline{0,N_1-1},\ j=\overline{0,N_2-1}.
\end{aligned} \tag{11}$$

III. Study of the Stability and Convergence of the Difference Scheme

Lemma. For any grid functions $u_{i+1/2,j,k}\in\Omega_x$, $\upsilon_{i,j+1/2,k}\in\Omega_y$, $\omega_{i,j,k+1/2}\in\Omega_z$ satisfying conditions (10), (11), the following identities hold:

$$\left(L^{(1)}_{1h}u_{i+1/2,j,k},u_{i+1/2,j,k}\right)=\left(L^{(2)}_{1h}\upsilon_{i,j+1/2,k},\upsilon_{i,j+1/2,k}\right)=\left(L^{(3)}_{1h}\omega_{i,j,k+1/2},\omega_{i,j,k+1/2}\right)=0 \tag{12}$$

where the summation is performed by internal nodes of the mesh $\Omega_x\cup\Omega_y\cup\Omega_z$.

We define the norm of the velocity vector as follows:

$$\left\|\vec{V}^n\right\|^2=\sum_{\Omega_x}(u^n_{i+1/2,j,k})^2h_1h_2h_3+\sum_{\Omega_y}(\upsilon^n_{i,j+1/2,k})^2h_1h_2h_3+\sum_{\Omega_z}(\omega^n_{i,j,k+1/2})^2h_1h_2h_3. \tag{13}$$

Multiplying the differential equations (1)-(3) by $2\tau u^{n+1}_{i+1/2,j,k}h_1h_2h_3$, $2\tau\upsilon^{n+1}_{i,j+1/2,k}h_1h_2h_3$ and $2\tau\omega^{n+1}_{i,j,k+1/2}h_1h_2h_3$ respectively, then summing them over points of $\Omega_x,\Omega_y,\Omega_z$, we obtain the following basic energy inequality:

$$\left\|V^{n+1}\right\|^2-\left\|V^n\right\|^2+\left\|V^{n+1}-V^n\right\|^2+2\tau(L_{1h}V^n,V^{n+1})+2\tau\left(\sum_{\Omega_x}p^{n+1}_{x_1}u^{n+1}_{i+1/2,j,k}+\right. \tag{14}$$
$$\left.+\sum_{\Omega_y}p^{n+1}_{x_2}\upsilon^{n+1}_{i,j+1/2,k}+\sum_{\Omega_z}p^{n+1}_{x_3}\omega^{n+1}_{i,j,k+1/2}\right)h_1h_2h_3+2\tau d_h\le\frac{2\tau}{De}|S_h|+2\tau\left|(\vec{f}^{\,h},V^{n+1})\right|.$$

Let us evaluate the term in equation (14). Considering the conditions (11), it can be seen that

$$d_h=\frac{\tau}{\mathrm{Re}_T}\left[\sum_{\Omega_x}\left(a_{i,j,k}u^n_{x_1,i+1/2,j,k}u^{n+1}_{x_1,i+1/2,j,k}+a_{i+1/2,j+1/2,k}u^n{}_{x_2,i+1/2,j,k}u^{n+1}_{x_2,i+1/2,j,k}+\right.\right. \tag{15}$$
$$\left.+a_{i+1/2,j,k+1/2}u^n_{x_3,i+1/2,j,k}u^{n+1}_{x_3,i+1/2,j,k}\right)h_1h_2h_3+$$
$$+\sum_{\Omega_y}\left(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,i+1/2,j,k}\upsilon^{n+1}_{x_1,i+1/2,j,k}+\right.$$
$$+a_{i,j+1,k}\upsilon^n{}_{x_2,i,j,k}\upsilon^{n+1}{}_{x_2,i,j,k}+$$
$$\left.+a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,i,j+1/2,k+1/2}\upsilon^{n+1}_{x_3,i,j+1/2,k+1/2}\right)h_1h_2h_3+$$

$$+\sum_{\Omega_z}\Big(a_{i+1/2,j,k+1/2}\omega^n_{x_1,i+1/2,j,k+1/2}\omega^{n+1}_{x_1,i+1/2,j,k+1/2}+$$
$$+a_{i,j+1/2,k}\omega^n_{x_2,i,j+1/2,k}\omega^{n+1}_{x_2,i,j+1/2,k}+a_{i,j,k+1}\omega^n_{x_3,i,j,k}\omega^{n+1}_{x_3,i,j,k}\Big)h_1h_2h_3\Big]$$

Using Young's inequality and the boundedness of the coefficient $a(x_{1i},x_{2j},x_{3k})$ from below, we have

$$|d_h|\ge C_1\left(\|\nabla_h\vec V^n\|^2+\|\nabla_h\vec V^{n+1}\|^2-\|\nabla_h(\vec V^{n+1}-\vec V^n)\|^2\right) \tag{16}$$

where

$$\|\nabla\vec V\|^2=\|\vec V_{x_1}\|^2+\|\vec V_{x_2}\|^2+\|\vec V_{x_3}\|^2, \tag{17}$$

$C_1=0{,}5a\tau/\mathrm{Re}_T$.

The term S_h can be written as follows:

$$S_h=\sum_{\Omega_x}\upsilon^n_{i+1/2,j,k}u^{n+1}_{i+1/2,j,k}h_1h_2h_3-\sum_{\Omega_y}u^n_{i,j+1/2,k}\upsilon^{n+1}_{i,j+1/2,k}h_1h_2h_3= \tag{18}$$
$$\sum_{\Omega_x}\upsilon^n_{i+1/2,j,k}(u^{n+1}_{i+1/2,j,k}-u^n_{i+1/2,j,k})h_1h_2h_3-\sum_{\Omega_y}u^n_{i,j+1/2,k}(\upsilon^{n+1}_{i,j+1/2,k}-\upsilon^n_{i,j+1/2,k})h_1h_2h_3.$$

Further, using Young's inequality, we have

$$|S_h|\le\left(\sum_{\Omega_y}(\upsilon^n_{i,j+1/2,k})^2h_1h_2h_3+\sum_{\Omega_x}(u^n_{i+1/2,j,k})^2h_1h_2h_3\right)+ \tag{19}$$
$$+\left(\sum_{\Omega_x}(u^{n+1}_{i+1/2,j,k}-u^n_{i+1/2,j,k})^2h_1h_2h_3+\sum_{\Omega_y}(\upsilon^{n+1}_{i,j+1/2,k}-\upsilon^n_{i,j+1/2,k})^2h_1h_2h_3\right).$$

Adding non-negative summands

$$\sum_{\Omega_z}(\omega^n_{i,j,k+1/2})^2h_1h_2h_3+\sum_{\Omega_z}(\omega^{n+1}_{i,j,k+1/2}-\omega^n_{i,j,k+1/2})^2h_1h_2h_3 \tag{20}$$

to the right-hand side of the inequality, we obtain

$$|S_h|\le\|\vec V^n\|^2+\|\vec V^{n+1}-\vec V^n\|^2. \tag{21}$$

By virtue the Lemma proved above, we have

$$2\tau(L_{1h}\vec V^n,\vec V^{n+1})=2\tau^2(L_{1h}\vec V^n,\vec V^n_t) \tag{22}$$

Using the Cauchy-Schwarz inequality we obtain:

$$\left|2\tau^2(L_{1h}V^n,V^n_t)\right|\le C_2\tau^2\left\{\sum_{\Omega_h}[(u^n_{i+1/2,j,k})^2+(\upsilon^n_{i,j+1/2,k})^2+(\omega^n_{i,j,k+1/2})]^2h_1h_2h_3\right\}^{1/2}\cdot\|\vec V^n_t\|=$$
$$=C_2\tau^2\left\||\vec V^n|^2\right\|\cdot\|\vec V^n_t\|. \tag{23}$$

The term $\left\||\vec V^n|^2\right\|$ is evaluated as follows [18]:

$$\left\||\vec V^n|^2\right\|\le\left(\frac{4}{3}\right)^{\frac{3}{4}}\|\vec V^n\|^{1/2}\|\nabla_h\vec V^n\|^{3/2}. \tag{24}$$

Then

$$\left|2\tau^2(L_{1h}\vec V^n,\vec V^n_t)\right|\le C_2(4/3)^{3/4}\tau^2\|\vec V^n\|^{1/2}\|\nabla_h\vec V^n\|^{3/2}\|\vec V^n_t\|\le\|\vec V^n_t\|^2+C_3\|\vec V^n\|\|\nabla_h\vec V^n\|^3 \tag{25}$$

where $C_3=\dfrac{2^{5/2}C_2\tau^2}{3^{3/4}}$.

$$\|\vec V^{n+1}\|^2-\|\vec V^n\|^2+\frac{1}{2}\|\vec V^{n+1}-\vec V^n\|^2+C_1\left(\|\nabla_h\vec V^n\|^2+\|\nabla_h\vec V^{n+1}\|^2-\|\nabla_h(\vec V^{n+1}-\vec V^n)\|^2\right)-C_3\|\vec V^n_t\|^2- \tag{26}$$
$$-C_3\|\vec V^n\|\|\nabla_h\vec V^n\|^3\le\frac{2\tau}{De}\left(\|\vec V^n\|^2+\|\vec V^{n+1}-\vec V^n\|^2\right)+2\tau|(\vec f^n,\vec V^{n+1})|$$

Hence we have

$$\vec V^{n+1\,2}+C_1\sum_{k=0}^n\nabla_h\vec V^{n+1\,2}\le V^{0\,2}+2\tau\left(\sum_{k=0}^n\vec f^k\right)\left(\vec V^0+2\tau\sum_{k=0}^n\vec f^k\right)\le \tag{27}$$
$$\le2\vec V^{0\,2}+5\left(\tau\sum_{k=0}^n\vec f^k\right)^2.$$

In order to study the convergence of the solution of finite difference problem to the solution of the differential problem, we consider the finite difference equations for the equations of the atmospheric boundary layer:

$$u^n_{t,h,i+1/2,j,k}+L^{(1)}_{1h}u^n_{x_1,h,i+1/2,j,k}+P^{n+1}_{x_1,h,i,j,k}=\frac{1}{De}\upsilon^n_{h,i,j+1/2,k}+$$
$$\frac{1}{\mathrm{Re}_T}\Big[(a_{i,j,k}u^n_{x_1,h,i+1/2,j,k})_{\bar x_1}+(a_{i+1/2,j+1/2,k}u^n_{x_2,h,i+1/2,j,k})_{\bar x_2}+ \tag{28}$$
$$+(a_{i+1/2,j,k+1/2}u^n_{x_3,h,i+1/2,j,k})_{\bar x_3}\Big]+f^0_{i+1/2,j,k},$$
$$i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},k=\overline{1,N_3-1}$$

$$\upsilon_{t,h,i,j+1/2,k}+L^{(2)}_{1h}\upsilon^n_{i,j+1/2,k}+P^{n+1}_{x_2,h,i,j,k}=-\frac{1}{De}u^n_{h,i+1/2,j,k}+$$
$$+\frac{1}{\mathrm{Re}_T}\Big[(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,h,i+1/2,j+1/2,k})_{\bar x_1}+(a_{i,j+1,k}\upsilon^n_{x_2,h,i,j,k})_{\bar x_2}+ \tag{29}$$
$$+(a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,h,i,j+1/2,k+1/2})_{\bar x_3}\Big]+f^0_{i,j+1/2,k},$$
$$i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},k=\overline{1,N_3-1}$$

$$\omega^n_{t,h,i,j,k+1/2}+L^{(3)}_{1h}\omega^n_{h,i,j,k+1/2}+P^{n+1}_{x_3,h,i,j,k}=\bar\lambda+$$
$$+\frac{1}{\mathrm{Re}_T}\Big[(a_{i+1/2,j,k+1/2}\omega^n_{x_1,h,i+1/2,j,k+1/2})_{\bar x_1}+(a_{i,j+1/2,k}\omega^n_{x_2,h,i,j+1/2,k})_{\bar x_2}+ \tag{30}$$
$$+(a_{i,j,k+1}\omega^n_{x_3,h,i,j,k})_{\bar x_3}\Big]+f^0_{i,j,k+1/2},$$
$$i=\overline{1,N_1-2},\ j=\overline{1,N_2-1},k=\overline{1,N_3-1}$$

$$u^{n+1}_{\bar x_1,h,i+1/2,j,k}+\upsilon^{n+1}_{\bar x_2,h,i,j+1/2,k}+\omega^{n+1}_{\bar x_3,h,i,j,k+1/2}=0 \tag{31}$$

with following initial and boundary conditions:

$$u^0_{i+1/2,j,k}=V^0(x_{1i}+0{,}5h_1,x_{2j},x_{3k}),\upsilon^0_{i,j+1/2,k}=V^0(x_{1i},x_{2,j}+0{,}5h_2,x_{3k}),\omega^0_{i,j,k+1/2}=V^0(x_{1i},x_{2j},x_{3k}+0{,}5h_3)$$
$$\upsilon^{n+1}_{0,j+1/2,k}=\upsilon^{n+1}_{N_1,j+1/2,k}=u^{n+1}_{1/2,j,k}=u^{n+1}_{N_1-1/2,j,k}=$$
$$=\omega^{n+1}_{0,j,k+1/2}=\omega^{n+1}_{N_1,j,k+1/2}=0,\ j=\overline{0,N_2-1},k=\overline{0,N_3-1},$$
$$\upsilon^{n+1}_{i,1/2,k}=\upsilon^{n+1}_{N_1,j-1/2,k}=u^{n+1}_{i+1/2,0,k}=u^{n+1}_{i+1/2,N_2,k}= \tag{32}$$
$$=\omega^{n+1}_{i,0,k+1/2}=\omega^{n+1}_{i,N_2,k+1/2}=0,\ i=\overline{0,N_1-1},k=\overline{0,N_3-1},$$
$$\upsilon^{n+1}_{i,j+1/2,0}=\upsilon^{n+1}_{i,j+1/2,N_3}=u^{n+1}_{i+1/2,j,0}=u^{n+1}_{i+1/2,j,N_3}=$$
$$=\omega^{n+1}_{i,j,1/2}=\omega^{n+1}_{i,j,N_3-1/2}=0,\ i=\overline{0,N_1-1},\ j=\overline{0,N_2-1},$$

Let us define the error of solutions of the differential problem (1)-(5) and the difference problem (28)-(32) as follows:

$$\varphi^{(1)n}_{i+1/2,j,k}=u^n_{h,i+1/2,j,k}-u^n_{i+1/2,j,k},$$
$$\varphi^{(2)n}_{i,j+1/2,k}=\upsilon^n_{h,i,j+1/2,k}-\upsilon^n_{i,j+1/2,k},$$
$$\varphi^{(3)n}_{i,j,k+1/2}=\omega^n_{h,i,j,k+1/2}-\omega^n_{i,j,k+1/2},$$
$$\pi^{n+1}_{i,j,k}=p^{n+1}_{h,i,j,k}-p^{n+1}_{i,j,k}. \tag{33}$$

Expressing $u^n_{h,i+1/2,j,k},\upsilon^n_{h,i,j+1/2,k},\omega^n_{h,i,j,k+1/2},P^{n+1}_{h,i,j,k}$ through $\varphi^{(1)n}_{i+1/2,j,k},\phi^{(2)n}_{i,j+1/2,k},\phi^{(3)n}_{i,j,k+1/2},\pi^n_{i,j,k}$ from (33) and substituting them into (28)-(31), we obtain

$$\varphi^{(1)n}_t+L^{(1)}_{1h}\varphi^{(1)n}+\pi^n_{i,j,k}=\frac{1}{De}\varphi^{(2)n}_{i+1/2,j,k}+\frac{1}{\mathrm{Re}_T}\Big[(a_{i,j,k}\varphi^{(1)n}_{x_1,i+1/2,j,k}+ \tag{34}$$
$$+(a_{i+1/2,j+1/2,k}\varphi^{(1)n}_{x_2,i+1/2,j,k})_{\bar x_2}+(a_{i+1/2,j,k+1/2}\varphi^{(1)n}_{x_3,i+1/2,j,k})_{\bar x_3}\Big]+A^{(1)}_{i+1/2,j,k}+\psi^{(1)}_{i+1/2,j,k},$$

$$\varphi^{(2)n}_t+L^{(2)}_{1h}\varphi^{(2)n}+\pi^n_{i,j,k}=-\frac{1}{De}\varphi^{(1)n}_{i,j+1/2,k}+\frac{1}{\mathrm{Re}_T}\Big[(a_{i+1/2,j+1/2,k}\varphi^{(2)}_{x_1,i+1/2,j+1/2,k})_{\bar x_1}+ \tag{35}$$
$$(a_{i,j+1,k}\varphi^{(2)n}_{x_2,i,j,k})_{\bar x_2}+(a_{i,j+1/2,k+1/2}\varphi^{(2)n}_{x_3,i,j+1/2,k+1/2})_{\bar x_3}\Big]+A^{(2)}_{i,j+1/2,k}+\psi^{(2)}_{i,j+1/2,k},$$

$$\varphi^{(3)n}_t+L^{(3)}_{1h}\varphi^{(2)n}+\pi^n_{i,j,k}=\bar\lambda+\frac{1}{\mathrm{Re}_T}\Big[(a_{i+1/2,j,k+1/2}\varphi^{(3)}_{x_1,i+1/2,j,k+1/2})_{\bar x_1}+ \tag{36}$$
$$+(a_{i,j+1/2,k}\varphi^{(3)n}_{x_2,i,j+1/2,k})_{\bar x_2}+(a_{i,j,k+1}\varphi^{(3)n}_{x_2,i,j,k})_{\bar x_3}\Big]+A^{(3)}_{i,j,k+1/2}+\psi^{(3)}_{i,j,k+1/2},$$

$$\varphi^{(1)(n+1)}_{\bar x_1,i+1/2,j,k}+\varphi^{(2)(n+1)}_{\bar x_2,i,j+1/2,k}+\varphi^{(3)(n+1)}_{\bar x_3,i,j,k+1/2}=0 \tag{37}$$

where error of approximation of the difference scheme (28)-(32) on the exact solution of the differential problem (1)-(5) is defined as

$$\psi^{(1)}_{i+1/2,j,k} = \frac{1}{De}\upsilon^n_{i,j+1/2,k} + \frac{1}{\mathrm{Re}_T}[(a_{i,j,k}u^n_{x_1,i+1/2,j,k})_{\bar{x}_1} + (a_{i+1/2,j+1/2,k}u^n_{x_2,i+1/2,j,k})_{\bar{x}_2} +$$
$$+(a_{i+1/2,j,k+1/2}u^n_{x_3,i+1/2,j,k})_{\bar{x}_3}] - u^n_{t,i+1/2,j,k} - L^{(1)}_{1h}u^n - P^n_{x_1,i,j,k} \quad (38)$$
$$\psi^{(2)}_{i,j+1/2,k} = -\frac{1}{De}u^n_{i+1/2,j,k}\frac{1}{\mathrm{Re}_T}[(a_{i+1/2,j+1/2,k}\upsilon^n_{x_1,i+1/2,j+1/2,k})_{\bar{x}_1} + (a_{i,j+1,k}\upsilon^n_{x_2,i,j,k})_{\bar{x}_2} +$$
$$+(a_{i,j+1/2,k+1/2}\upsilon^n_{x_3,i,j+1/2,k+1/2})_{\bar{x}_3}] - \upsilon^n_{t,i,j+1/2,k} - L^{(2)}_{1h}\upsilon^n - P^n_{x_2,i,j,k}$$
$$\psi^{(3)}_{i,j,k+1/2} = \bar{\lambda} + \frac{1}{\mathrm{Re}_T}[(a_{i+1/2,j,k+1/2}\omega^n_{x_1,i+1/2,j,k+1/2})_{\bar{x}_1} + (a_{i,j+1/2,k}\omega^n_{x_2,i,j+1/2,k})_{\bar{x}_2} +$$
$$+(a_{i,j,k+1}\omega^n_{x_3,i,j,k})_{\bar{x}_3}] - \omega^n_{t,i,j,k+1/2} - L^{(3)}_{1h}\omega^n - P^n_{x_3,i,j,k}$$

and it has the second order of approximation by h and the first order by τ [19].

The initial and boundary conditions of the problem for errors (34)-(37) are defined as follows:

$$\phi^{(1)0}_{i+1/2,j,k} = 0, \phi^{(2)0}_{i,j+1/2,k} = 0, \phi^{(3)0}_{i,j,k+1/2} = 0$$
$$\varphi^{(2)n+1}_{0,j+1/2,k} = \varphi^{(2)n+1}_{N_1,j+1/2,k} = \varphi^{(1)n+1}_{1/2,j,k} = \varphi^{(1)n+1}_{N_1-1/2,j,k} = \varphi^{(3)n+1}_{0,j,k+1/2} = \varphi^{(3)n+1}_{N_1,j,k+1/2} = 0,$$
$$j = \overline{0, N_2-1}, k = \overline{0, N_3-1}$$
$$\varphi^{(2)n+1}_{i,1/2,k} = \varphi^{(2)n+1}_{N_1,j-1/2,k} = \varphi^{(1)n+1}_{i+1/2,0,k} = \varphi^{(1)n+1}_{i+1/2,N_2,k} = \varphi^{(3)n+1}_{i,0,k+1/2} = \varphi^{(3)n+1}_{i,N_2,k+1/2} = 0, \quad (39)$$
$$i = \overline{0, N_1-1}, k = \overline{0, N_3-1}$$
$$\varphi^{(2)n+1}_{i,j+1/2,0} = \varphi^{(2)n+1}_{i,j+1/2,N_3} = \varphi^{(1)n+1}_{i+1/2,j,0} = \varphi^{(1)n+1}_{i+1/2,j,N_3} = \varphi^{(3)n+1}_{i,j,1/2} = \varphi^{(3)n+1}_{i,j,N_3-1/2} = 0,$$
$$i = \overline{0, N_1-1}, j = \overline{0, N_2-1}$$

Multiplying the differential equation (34)-(37) by $2\tau\varphi^{(1)(n+1)}_{i+1/2,j,k}h_1h_2h_3$, $2\tau\varphi^{(2)(n+1)}_{i,j+1/2,k}h_1h_2h_3$, $2\tau\varphi^{(3)(n+1)}_{i,j,k+1/2}h_1h_2h_3$ respectively, then summing by grid domains $\Omega_x, \Omega_y, \Omega_z$, we obtain

$$\vec{\varphi}^{n+1\,2} - \left(1 + \frac{5\tau}{2}C_4 - \frac{2\tau}{De}\right)\vec{\varphi}^{n\,2} + (1-2\tau C_4)\nabla_h\vec{\varphi}^{n+1\,2} +$$
$$+\left(1 - \frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De}\right)\vec{\varphi}^{n+1} - \vec{\varphi}^{n\,2} + \quad (40)$$
$$+2\tau\left(C_1 - C_3\vec{\varphi}^n\nabla_h\vec{\varphi}^n\right)\nabla_h\vec{\varphi}^{n\,2} \le 2\tau\vec{\psi}^n\vec{\varphi}^{n+1}.$$

Let us denote $a = 1; b = 1 + \frac{5\tau}{2}C_4 - \frac{2\tau}{De}$, and rewrite (40) as follows:

$$a\left\|\vec{\phi}^{n+1}\right\|^2 - b\left\|\vec{\phi}^n\right\|^2 + \left(1 - \frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De}\right)\left\|\vec{\phi}^{n+1} - \vec{\phi}^n\right\|^2 + (1-2\tau C_4)\left\|\nabla_h\vec{\phi}^{n+1}\right\|^2 + \quad (41)$$
$$+2\tau\left(C_1 - C_3\left\|\vec{\phi}^n\right\|\left\|\nabla_h\vec{\phi}^n\right\|\right)\left\|\nabla_h\vec{\phi}^n\right\|^2 \le 2\tau\left\|\vec{\psi}^n\right\|\left\|\vec{\phi}^{n+1}\right\|$$

Let $a \ge b$. Then it follows that $\frac{2}{De} - \frac{5}{2}C_4 \ge 0.$ (42)

Let $C_1 - C_3\left\|\vec{\phi}^n\right\|\left\|\nabla_h\vec{\phi}^n\right\| \ge 0$; $1 - \frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De} > 0$; $1 - 2\tau C_4 > 0.$ (43)

Then considering that the third and fifth terms in the left-hand side of (41) are nonnegative, we obtain

$$a\left(\left\|\vec{\phi}^{n+1}\right\|^2 - \left\|\vec{\phi}^n\right\|^2\right) + C_5\left\|\nabla_h\vec{\phi}^{n+1}\right\|^2 \le 2\tau\left\|\vec{\psi}^n\right\|\left\|\vec{\phi}^{n+1}\right\| \quad (44)$$

where $C_5 = 1 - 2\tau C_4$.

Considering that $a = 1$, we have

$$\left\|\vec{\phi}^{n+1}\right\|^2 - \left\|\vec{\phi}^n\right\|^2 + C_5\left\|\nabla_h\vec{\phi}^{n+1}\right\|^2 \le 2\tau\left\|\vec{\psi}^n\right\|\left\|\vec{\phi}^{n+1}\right\|.$$

Considering similarly as for problem (1)-(5), we obtain the following estimate for the problem (34)-(37), (39):

$$\left\|\vec{\phi}^{n+1}\right\|^2 + C_5\sum_{k=0}^{n}\left\|\nabla_h\vec{\phi}^{n+1}\right\|^2 \le 5\tau^2\left(\sum_{k=0}^{n}\left\|\vec{\psi}^k\right\|\right)^2, \quad (45)$$

Further, considering that $\vec{\psi}^n = O(h^2)$ according to (38), we finally have

$$\left\|\vec{\phi}^{n+1}\right\|^2 + C_5\sum_{k=0}^{n}\left\|\nabla_h\vec{\phi}^{n+1}\right\|^2 \le C_6(\tau^2 + h^4) \quad (46)$$

which proves the convergence of the solution of the difference problem (28)-(32) to the solution of the differential problem (1)-(5).

Theorem. Let the conditions (43) hold. Then the difference scheme (28)-(32) is stable and its solution converges to the solution of the differential problem (1)-(5) with the speed determined by the inequality (46).

IV. Numerical Calculations

The numerical calculations with different values of the input parameters were performed based on the model and proposed algorithm described above. The considering area is 35x35 km^2, and the height of the surface layer is 3500 m. The convection parameter $\lambda = 0{,}16\ m/(s\cdot°C)$. The stratification parameter S in terms of physical meaning determine the temperature variations with altitude; therefore the calculations were performed based on the vertical temperature gradient. Carioles force is equal to $l = 10^{-4}s^{-1}$.

The values of the horizontal and vertical turbulent exchange coefficients were taken as follows: $\mu_x = \mu_y = 6\cdot 10^3 m^2/s$, $\nu = 30\ m^2/s$.

The characteristic scale of length, the wind speed and temperature are set as follows: $L = 35000$ m, $U^* = 10$ m/s, $\theta^* = 20°C$.

The following formulas were used for determination of dimensionless values of the input parameters λ, l, S, μ_x, μ_y, X, Y, H:

$$\bar{\lambda} = \frac{\lambda\cdot\theta^*}{U^*},\ \bar{l} = \frac{l\cdot L}{U^*},\ \bar{S} = \frac{S\cdot L}{\theta^*},\ \bar{\mu}_x = \frac{\mu}{L\cdot U^*},\ \bar{\mu}_y = \frac{\mu}{L\cdot U^*},$$
$$\bar{X} = \frac{X}{L},\ \bar{Y} = \frac{Y}{L},\ \bar{H} = \frac{H}{L}$$

where U^* is characteristic velocity, θ is characteristic temperature, L is the length scale.

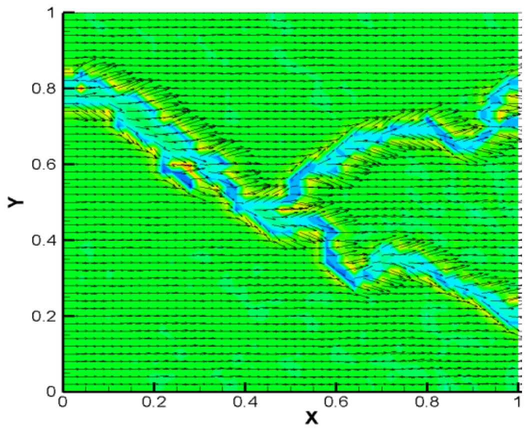

Fig. 1. The deviation of wind direction above the water surfaces at a wind speed equal to 1 m/s

Figures 1 and 2 show the wind deflection over the water surfaces. This process is observed at moderate wind speeds and it is difficult to see it at high speeds of the fluctuations wind over water surfaces.

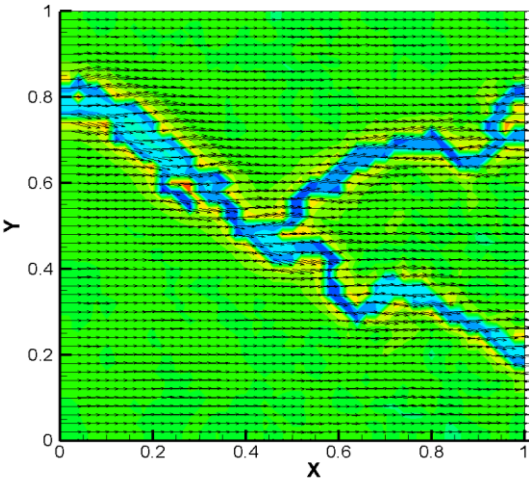

Fig. 2. The deviation of wind direction above the water surfaces at a wind speed equal to 2 m/s

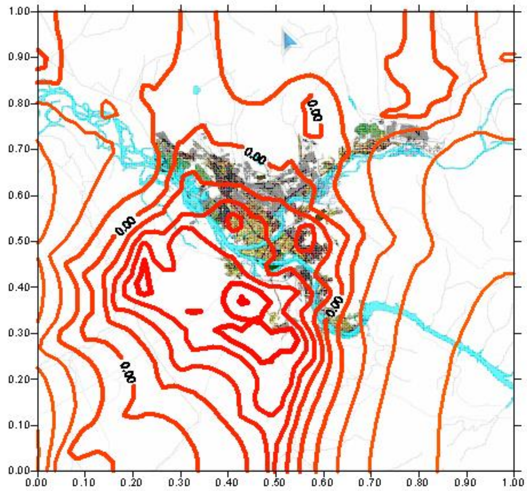

Fig. 3. Isolines of CO (carbon monoxide) distribution under the unstable weather conditions (in the absence of wind)

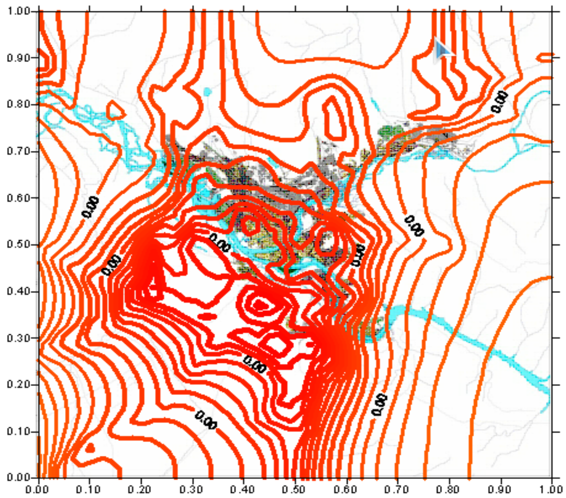

Fig. 4. Isolines of CO_2 (carbon dioxide) distribution under the unstable weather conditions (in the absence of wind)

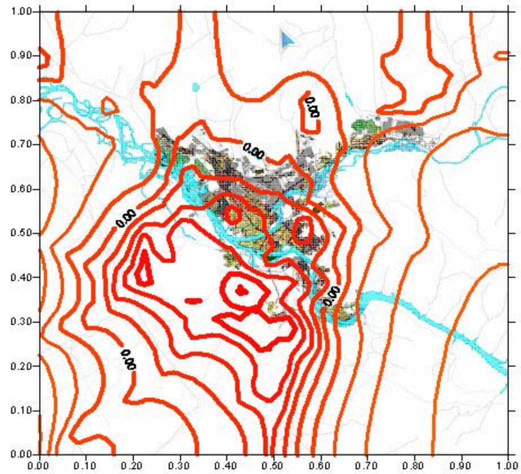

Fig. 5. Isolines of CO (carbon monoxide) distribution at the west wind direction at a speed equal to 1 m/s

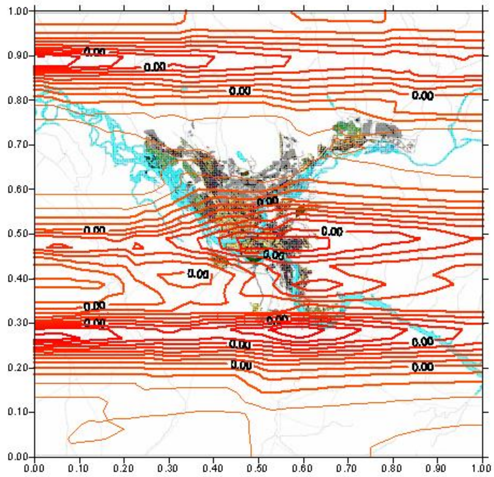

Fig. 6. Isolines of the distribution of CO_2 (carbon dioxide) in the west wind direction at a speed equal to 2 m/s

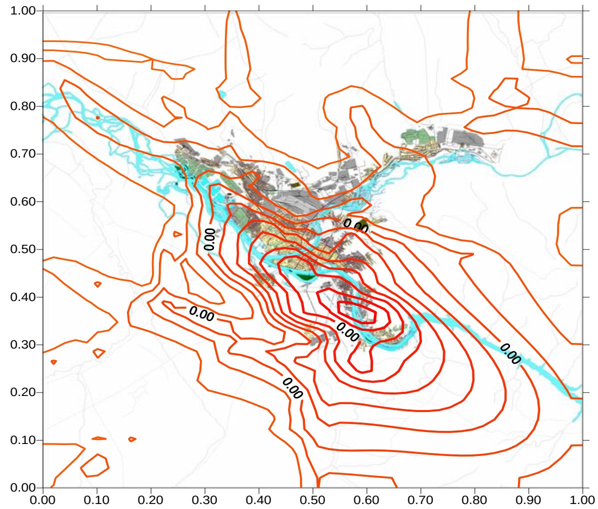

Fig. 7. The isolines of CO distribution (carbon monoxide) in the north-west direction of the wind at a speed equal to 6 m/s

The source of carbon monoxide CO is the gas emission of cars. It is formed by the combustion of fuel in the internal combustion engines at insufficient temperatures [20]-[22]. Under the natural conditions, CO carbon monoxide is formed on the surface of the earth during the incomplete decomposition of organic compounds and the combustion of biomass, mainly during the forest fires. Figures 3 and 4 show the distribution isolines of this substance over the city of Ust-Kamenogorsk in the absence of wind.

Figures 5, 6 and 7 show the isolines of contaminant distribution in the west and north-west wind directions.

As a result of numerical experiments, it was established that the anthropogenic impurity produced by industrial enterprises and picked up by wind currents at different directions, moves to large distances depending on the wind speed, which leads to the imposition of pollution fields. At unfavorable metaconditions, anthropogenic impurity forms a cloud over an industrial city.

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