INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2020, VOL. 66, NO. 3, PP. 589-594
Manuscript received February 11, 2020; revised July, 2020. DOI: 10.24425/ijet.2020.134016

Filtering Property of Signal Sampling in General and Under-Sampling as a Specific Operation of Filtering Connected with Signal Shaping at the Same Time

Andrzej Borys

***Abstract*—In this paper, we show that signal sampling operation can be considered as a kind of all-pass filtering in the time domain, when the Nyquist frequency is larger or equal to the maximal frequency in the spectrum of a signal sampled. We demonstrate that this seemingly obvious observation has wide-ranging implications. They are discussed here in detail. Furthermore, we discuss also signal shaping effects that occur in the case of signal under-sampling. That is, when the Nyquist frequency is smaller than the maximal frequency in the spectrum of a signal sampled. Further, we explain the mechanism of a specific signal distortion that arises under these circumstances. We call it the signal shaping, not the signal aliasing, because of many reasons discussed throughout this paper. Mainly however because of the fact that the operation behind it, called also the signal shaping here, is not a filtering in a usual sense. And, it is shown that this kind of shaping depends upon the sampling phase. Furthermore, formulated in other words, this operation can be viewed as a one which shapes the signal and performs the low-pass filtering of it at the same time. Also, an interesting relation connecting the Fourier transform of a signal filtered with the use of an ideal low-pass filter having the cut frequency lying in the region of under-sampling with the Fourier transforms of its two under-sampled versions is derived. This relation is presented in the time domain, too.**

***Keywords*—Signal sampling, filtering, discrete-time Fourier transform**

I. Introduction

As far as sampling of signals, sampling theorem, and reconstruction formula are concerned, it seems that everything has been already fully explained and understood. Maybe? Although in this paper, we rather doubt it. For example, it seems to us that not all electrical engineers realize that the signal sampling operation is itself a filter and/or a signal shaping device. And that it has some intriguing consequences, which can allow better understanding of the fundamentals of digital signal processing. As, for example, the fact that the discrete-time Fourier transform (DTFT) is not a periodic function (transformation). What is often taught to be the DTFT [1-7] is simply a periodic replicating this part of the DTFT which lies in the range between negative and positive values of the Nyquist frequency [4] - on the whole frequency axis. So, obviously, these two things do not mean the same.

Andrzej Borys is with the Department of Marine Telecommunications, Faculty of Electrical Engineering, Gdynia Maritime University, Gdynia, Poland (e-mail: a.borys@we.umg.edu.pl).

In this paper, we can distinguish two parts. The first part is devoted to presentation of the filtering property of the signal sampling operation in the case when the Nyquist frequency is larger or equal to the maximal frequency in the spectrum of a signal sampled. Whereas the second one discusses what happens when the condition mentioned above is not satisfied. Note that the latter case is much more complicated and more difficult to understand than the first one because it encompasses also the effect which is called signal aliasing. Then, as we show in this paper, the signal sampling can be described as a specific operation containing both the filtering and signal shaping

The remainder of this paper is organized as follows. In the next section, we present a definition of the notion of a signal object. The third section shows that the signal sampling behaves as an all-pass signal filtering operation in the case when the Nyquist frequency is larger or equal to the maximal frequency in the spectrum of a signal sampled. The behavior of the signal sampling viewed as its shaping is a subject of the fourth section. Such behavior occurs when the Nyquist frequency is smaller from the maximal frequency in the spectrum of a signal sampled. The paper ends with some concluding remarks.

II. Notion of a Signal Object

We introduce this notion here because it will allow us to define transparently the operation of sampling of signals as a kind of a filter and/or of a signal shaper at the same time. To this end, consider a signal $x(t)$ of a continuous time variable t and denote the maximal frequency present in its spectrum by f_m. So, this signal can be sampled and reconstructed perfectly if the sampling period T fulfils the following:

$$1/T = f_s \geq 2 f_m , \tag{1}$$

where f_s means the corresponding sampling frequency. The reconstruction formula to be used has the form

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\mathrm{sinc}(t/T - k) = \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}(t/T - k) . \tag{2}$$

In (2), $x(kT) = x[k], k = ...-1,0,1,...,$ mean the samples of the signal $x(t)$ sampled with the frequency f_s, whereas the function $\mathrm{sinc}(t)$ is defined as

$$\mathrm{sinc}(t) = \sin(\pi t)/\pi t \quad \text{for } t \neq 0 \quad \text{and} \quad 1 \text{ for } t = 0 . \tag{3}$$

Further, let us use here the symbol $x(k) = x(kT) = x[k]$, which was used above to denote the sequence of signal sampled values, to denote also the signal x of the discrete time variable $k \to kT$, containing elements of this sequence. Obviously, the local meaning of the above symbol will follow from the context of place, where it is actually used. Example usage of the symbols $x(t)$ and $x(k)$ is visualized in Fig. 1.

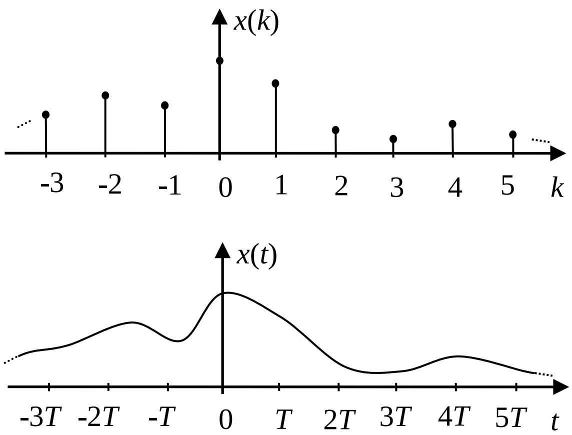

Fig. 1. Example discrete-time signal (upper curve), where the integers $...,-1,0,1,...$ mean successive values of discrete time variable *k*, and equivalent signal in continuous time domain (lower curve), where *t* stands for continuous time variable. Figure taken from [8].

Note now that by virtue of the sampling theorem the signals shown in Fig. 1, which are represented there by the lower and upper curves, can be perceived as being equivalent. They are equivalent because they can be obtained from each other via the sampling fulfilling (1) and the reconstruction formula (2), where the latter regards the reverse direction. Further, observe that we can choose any value of *T* from the range $\left(0,1/(2f_m)\right\rangle$ to perform sampling of the signal $x(t)$. As a result, we get an infinite number of discrete signals $x(k)$ that will be equivalent to the signal $x(t)$. So, all these discrete signals $x(k)$ together with their "source" signal $x(t)$ can be perceived as a one object. And, after [8], we can call it a signal object. This object will be a specific object consisting of an infinite number of elements having such a property that each of them is only an image of all the others, in the sense defined above.

As to the details of a concept of the signal object, the interested reader is referred to [8]. Moreover, in what follows, we will use a notation $x(t,k)$ for denoting a signal object containing the whole family of signals $x(k)$ associated with $x(t)$, including also $x(t)$.

III. Signal Sampling as an All-Pass Filtering When Conditions of Sampling Theorem Are Satisfied

The facts, which we shall be discussing now, are well known, but are not interpreted in the way as we will do here. Some might say that our interpretation of the signal sampling operation presented in this paper is irrelevant. On the contrary, we show that nothing could be more wrong. Here, we focus only on a one example to show the strength of our interpretation. Thanks to it we will able to derive in a simple way correct expressions for the DTFT.

The signal sampling operation within the range $\left(0,1/(2f_m)\right\rangle$ of the sampling periods *T* is nothing else than a kind of all-pass filtering. Why? After performing sampling of a signal $x(t)$ with a sampling period lying in the range mentioned above, we get its discrete version $x(k)$. But, when applying the inverse operation to $x(k)$ given by (2), we recover a perfect version of $x(t)$. Therefore, the sampling of a signal can be interpreted as such a manipulation on this signal that essentially does not change that signal. Obviously, the above manipulation changes an "external" image of the signal considered, from a one of a continuous time to that which is its discretized version, with a concrete *T* chosen from an infinite number of values mentioned above. In other words, the above manipulation can be perceived as an operation that takes a one element of a given signal object $x(t,k)$ (as defined in the previous section) and brings another one of the same signal object as a result. So, in this function, it perfectly resembles the behavior of an all-pass filter.

On the other hand, when the frequency domain is considered, the signal sampling in the range $\left(0,1/(2f_m)\right\rangle$ of sampling periods *T* or equivalently in the range $\left\langle 2f_m,\infty\right)$ of sampling frequencies $f_s = 1/T$ behaves as an ideal low-pass filter. That is it behaves as a filter that possesses the transfer function equal identically to 1 in the range of frequencies $\left\langle -f_s/2, f_s/2\right\rangle$, but having identically the zero value outside this range. This is very apparent when we calculate the Fourier transform of $x(t)$ using (2). So, applying the definition of the Fourier transform (denoted symbolically by $\mathrm{F}\,(\cdot)$) to both sides of (2), we get

$$\begin{aligned} \mathrm{F}\,\left(x(t)\right) &= \mathrm{F}\,\left(\sum_{k=-\infty}^{\infty} x[k]\,\mathrm{sinc}(t/T-k)\right) = \\ &= \sum_{k=-\infty}^{\infty} x[k]\,\mathrm{F}\,\left(\mathrm{sinc}(t/T-k)\right) . \end{aligned} \tag{4}$$

Further, taking into account in (4) the fact that the Fourier transform of the function $\text{sinc}(t/T-k)$ is given by

$$\text{F}\left(\text{sinc}(t/T-k)\right)=T\cdot\text{rect}(fT)\exp(-j2\pi fkT), \quad (5)$$

where the function $\text{rect}(x)$ means the following:

$$\text{rect}(x)=1 \text{ for } |x|\le\frac{1}{2} \text{ and } 0 \text{ for } |x|>\frac{1}{2}, \quad (6)$$

we finally arrive at

$$\begin{aligned}\text{F}\left(x(t)\right)&=T\sum_{k=-\infty}^{\infty}x[k]\,\text{rect}(fT)\exp(-j2\pi fkT)=\\&=\frac{1}{f_s}\sum_{k=-\infty}^{\infty}x[k]\,\text{rect}\left(\frac{f}{f_s}\right)\exp\left(-j2\pi k\frac{f}{f_s}\right)\ .\end{aligned} \quad (7)$$

Note that a further simplification of (7) is possible when we take into account (6) in (7). Then, we get from the latter the following:

$$\begin{aligned}\text{F}\left(x(t)\right)&=\sum_{k=-\infty}^{\infty}\bar{x}[k]\exp(-j2\pi fkT)=\\&=\sum_{k=-\infty}^{\infty}\bar{x}[k]\exp\left(-j2\pi k\frac{f}{f_s}\right) \text{ for } \left|\frac{f}{f_s}\right|\le\frac{1}{2} \text{ and}\\\text{F}\left(x(t)\right)&\equiv 0 \text{ for } \left|\frac{f}{f_s}\right|>\frac{1}{2},\end{aligned} \quad (8)$$

where signal samples defined as $\bar{x}[k]=T\cdot x[k]$ are used.

It is clear from (8) that the right-hand side expression in the first line of (8), which is used to define the DTFT [1-7], is nothing else than the usual Fourier transform of an un-sampled signal $x(t)$. This is however valid only for the region of frequencies f fulfilling $|f|\le f_s/2$. Outside this range, the expression $\sum_{k=-\infty}^{\infty}\bar{x}[k]\exp(-j2\pi fkT)$ replicates $\text{F}\left(x(t)\right)$ infinitely many times, what can be easily deduced from (8), too. So, these two observations lead us to draw the following conclusion: exclusively (8) should be used as the definition of DTFT. The function $\sum_{k=-\infty}^{\infty}\bar{x}[k]\exp(-j2\pi fkT)$ of a variable f allowed to assume any value on the frequency axis will be then a periodic function, as stated in the literature [1-7]. And, it can be perceived as a "periodic extension" of the DTFT defined by (8).

Let us summarize now some other important conclusions which follow from the above considerations. First, (8) is an equivalent of (2) in the frequency domain. So, we can call it a perfect signal reconstruction formula in the frequency domain. It allows to calculate directly the Fourier transform of a signal being a member of a corresponding signal object which is given only by its samples. Thereby, we omit then a calculation that uses the Fourier integral. Second, note that in our formulation of the DTFT we do not need to operate with sampled signals represented by curves involving Dirac impulses as shown, for example, in Fig. 2.

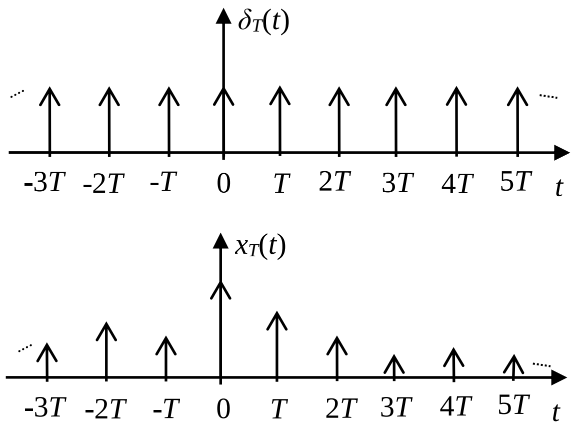

Fig. 2. Visualization of Dirac comb signal (top) and graphical representation of signal $x(t)$ multiplied by Dirac comb (bottom).

A way commonly used in modelling of the signal sampling process involves the use of the so-called Dirac comb $\delta_T(t)$ defined as

$$\delta_T(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)\ , \quad (9)$$

where $\delta(t)$ means the so-called Dirac impulse, which is not a usual function, but distribution. And, the Dirac comb is visualized in Fig. 2 by the upper curve. So, then, the sampled signal is assumed to have the form shown by the lower curve of Fig. 2. Mathematically, this sampled signal, we call here $x_T(t)$, is a multiplication of its continuous time version by $\delta_T(t)$. That is $x_T(t)=x(t)\cdot\delta_T(t)$.

Note that the above kind of modelling of the signal sampling process is a little bit cumbersome and artificial because it uses non-physical objects like Dirac impulses and Dirac comb. Unlike to that our approach avoids this. It is a natural way and obviously this is its advantage.

Thirdly, observe that the signals depicted in Fig. 1 (upper curve) and in Fig. 2 (lower curve) differ from each other not only graphically. They have also different spectra (Fourier transforms). Therefore, they belong to two different signal objects $x(t,k)$.

And fourth, see also that we do not need to use the notion of a Fourier series and the "periodic extension" of the DTFT mentioned before to obtain a definition of the inverse DTDF (that is the IDTFT) in our approach. It naturally derives here from applying a standard inverse Fourier transform to the DTFT given by (8). That is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{F}\left(x(t)\right)\exp\left(j2\pi kT\right)df = \\
&= \int_{-f_s/2}^{f_s/2} \mathrm{F}\left(x(t)\right)\exp\left(j2\pi fkT\right)df = \\
&= \sum_{n=-\infty}^{\infty} \overline{x}\left[n\right] \int_{-f_s/2}^{f_s/2} \exp\left(j2\pi fT\left(k-n\right)\right)df = \\
&= \overline{x}\left[n=k\right]\cdot f\Big|_{-f_s/2}^{f_s/2} = \overline{x}\left[k\right]\cdot\left(\frac{1}{2T}-\left(-\frac{1}{2T}\right)\right) = \\
&= \frac{\overline{x}\left[k\right]}{T} = x\left[k\right] = x\left(kT\right) \quad .
\end{aligned} \tag{10}$$

In derivation of (10), we have used the result given by (8), the fact that $f_s = 1/T$, and some other defining equations introduced before. We have also used the following:

$$\begin{aligned}
&\int_{-f_s/2}^{f_s/2} \exp\left(j2\pi fT\left(k-n\right)\right)df = \frac{1}{j2\pi T\left(k-n\right)}\cdot \\
&\cdot\exp\left(j2\pi fT\left(k-n\right)\right)\Big|_{-f_s/2}^{f_s/2} = 0 \quad \text{for } k\neq n \quad .
\end{aligned} \tag{11}$$

IV. Signal Sampling Viewed as Its Shaping

Let us start this section with recalling the condition under which the signal sampling operation can be considered as a kind of pure all-pass filtering as shown in the previous section. This holds when the sampling operation on a signal $x(t)$ is carried out with a sampling frequency f_s fulfilling (1). Let us rewrite however (1) for the needs of this section with a slightly modified notation as follows:

$$1/T_s = f_s \geq 2 f_{mx}, \tag{12}$$

where f_{mx} means the maximal frequency present in the spectrum of the signal considered and $T_s = 1/f_s$ denotes the sampling period related with f_s. Obviously, if (12) is not satisfied, then there are frequencies in the signal spectrum for which the so-called aliasing effect occurs [1], [4], [6]. We call here this effect a signal shaping. Why? The reasons for this will be derived and presented in this section.

Further, note that we introduce here the notion of a signal shaping because it will allow us to define transparently the operation of signal sampling as a kind of signal filtering and its shaping at the same time. And, we will also show that these two processes depend upon the segment of the signal spectrum considered at a given moment.

Now, to explain the mechanism of this combined behavior, let us first choose the following value: $f_s = 3 f_{mx} \geq 2 f_{mx}$ of the sampling frequency of the signal $x(t)$ mentioned above.

So, in this case, a perfect reconstruction of the signal $x(t)$ is possible and it will have the following form:

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} x\left(kT_s\right)\,\mathrm{sinc}\left(t/T_s - k\right) = \\
&= \sum_{k=-\infty}^{\infty} x\left[kT\right]\,\mathrm{sinc}\left(t/T - k\right)
\end{aligned} \tag{13}$$

with $T_s = 1/f_s = 1/(3 f_{mx}) = T$ and the function $\mathrm{sinc}(x)$ defined in (3).

But, second, if we choose the sampling frequency to be equal to $f_s = 3 f_{mx}/2 < 2 f_{mx}$, then the condition (12) will be violated. In this case, we will have to do with the so-called aliasing effect [1], [4], [6], and obviously the usage of the reconstruction formula as the one used in (13) with the doubled value of $T_s = 2T$ will not be possible.

To proceed further, let us now divide the sequence of the signal samples $x(kT) = x[k],\ k = ..., -2, -1, 0, 1, 2, ...,$ into two subsequences gathering separately all of them having even indices and those which possess odd ones. Definition and scheme illustrating how to build them up is shown below.

$$\begin{aligned}
&\{x(kT)\} = \{...\ x(-2T)\ \ x(-T)\ \ x(0)\ \ x(T)\ \ x(2T)\ \ x(3T)\ ...\} \\
&\{x'(kT)\} = \{...\ x(-2T)\ \ \ \ \ \ \ x(0)\ \ \ \ \ \ \ x(2T)\ \ \ \ \ \ \ ...\} \\
&\{x''(kT)\} = \{...\ \ \ \ \ \ \ x(-T)\ \ \ \ \ \ \ x(T)\ \ \ \ \ \ \ x(3T)\ ...\}
\end{aligned} \tag{14}$$

From (14), it follows that the following relation:

$$\{x(kT)\} = \{x'(kT)\} + \{x''(kT)\} \tag{15}$$

holds for the sequences $\{x(kT)\}$, $\{x'(kT)\}$, and $\{x''(kT)\}$ defined in (14). Further, the latter two sequences can be put into another form as follows:

$$\begin{aligned}
&\{x'(kT)\} = \{x(k_e T)\} = \{x(2nT)\} = \\
&= \{x_e(n\cdot 2T)\} = \{x_e[n]\}
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\{x''(kT)\} = \{x(k_o T)\} = \{x(2nT+T)\} = \\
&= \{x_o(n\cdot 2T)\} = \{x_o[n]\} \ ,
\end{aligned} \tag{17}$$

where $k_e = 2n$ and $k_o = 2n+1$ mean the corresponding even and odd integers, respectively, with $n \in \mathrm{Z}$, where Z denotes the whole set of integers. Also, it is worthy to see that the sequences $\{x_e(\cdot)\} = \{x_e[\cdot]\}$ and $\{x_o(\cdot)\} = \{x_o[\cdot]\}$ formulated in (16) and (17) can be considered as two possible sets of samples chosen from an infinite number of possible ones which can be achieved by sampling the signal $x(t)$ with the sampling period $T_s = 2T$. And, this observation will be utilized later.

Consider now the sequence $\{x_e(\cdot)\}=\{x_e[\cdot]\}$, but in isolation from any relation with the signal $x(t)$. Knowing that the elements of this sequence lie in the distance of $2T$ from each other, we can connect them through an interpolating function. And, one of the possible choices for doing this is the use of an interpolating function of the form which has that in (13). So, if we decide to do it this way, we get

$$y(t)=\sum_{k=-\infty}^{\infty} x_e[k]\,\mathrm{sinc}\big(t/(2T)-k\big)\ , \tag{18}$$

where $y(t)$ denotes an interpolating function obtained in the procedure described above. In the next step, let us calculate the Fourier transform of the function $y(t)$ given by (18). Applying directly the result presented in (8), we obtain

$$\begin{aligned}&\mathrm{F}\,\big(y(t)\big)=\sum_{k=-\infty}^{\infty}\bar{x}_e[k]\exp(-j2\pi kf\,2T)\\ &\quad\text{for}\quad |f|\le 1/(4T)\\ &\text{and}\\ &\mathrm{F}\,\big(y(t)\big)\equiv 0\quad\text{for}\ |f|>1/(4T)\ ,\end{aligned} \tag{19}$$

where $\bar{x}_e[k]$ means the following: $\bar{x}_e[k]=2T\cdot x_e[k]$. This result indicates that the function $y(t)$ represents a low-pass signal having the maximal frequency $f_{my}=1/(4T)$. Next, interpreting the distance $2T$ between the elements of the sequence $\{x_e(\cdot)\}=\{x_e[\cdot]\}$ as the sampling period used for sampling the interpolating function $y(t)$, we can reconstruct this function from its samples. This reconstruction given by (18) is perfect because the sampling frequency in this case satisfies (12), where f_{mx} occurring there should be replaced now by f_{my}. That is we have now $f_s=1/(2T)\ge 2f_{my}=$ $=2/(4T)=1/(2T)$.

Let us return now to the signal $x(t)$ and consider its relation with the signal $y(t)$. In view of that what was said above, the latter represents the former as its distorted version, with distortion caused by the so-called aliasing effect [1], [4], [6]. However, as we saw above, it is hard to see any aliasing products in the under-sampled signal. That is in $y(t)$. It is rather a kind of filtering of the signal $x(t)$ which leads to receiving a low-pass signal with the maximal frequency $f_{my}=(1/2)\cdot f_s$. But, as we will show in what follows, this kind of filtering is a special one because apart from performing filtering out all the signal spectrum components at frequencies $f>(1/2)\cdot f_s$ it shapes the signal at the same time.

So, because of this reason, this operation of signal under-sampling is called here signal shaping. This name seems to be more proper than the term signal aliasing. In particular that in terms of the notion of signal objects [8] $x(t)$ and $y(t)$ represent two different objects $x(t,k)$ and $y(t,k)$, respectively. For more details on this topic, see [8].

We notice also that by taking into account the sequence $\{x_o(\cdot)\}=\{x_o[\cdot]\}$ instead of the sequence $\{x_e(\cdot)\}=\{x_e[\cdot]\}$ in our considerations presented above we would arrive at the same conclusions. However, we would then work with a rather different interpolating function, say, $z(t)$. Why? Because it is rather hard to expect the sequences mentioned above to be identical. For more details regarding this fact, see [9].

Further, note that the sequence $\{x_o(\cdot)\}=\{x_o[\cdot]\}$ is shifted on the time axis by T with respect to the sequence $\{x_e(\cdot)\}=\{x_e[\cdot]\}$. And, if we assume that the sampling period $2T$ corresponds to a phase of 2π radians, the shift by T will correspond, respectively, to a phase of π radians. So, using this phase-oriented interpretation connected with the previous observation, we can express the latter shortly in this way: signal shaping effects (or equivalently aliasing effects in the terminology that is used) associated with the signal under-sampling are sampling phase dependent.

Now, we will concentrate on showing that really the signal under-sampling operation is not the usual kind of low-pass filtering. And, to this end, let us take into account the signal $x(t)$ once again and an ideal low-pass filter having the following transfer function:

$$\begin{aligned}&H(f)=1\quad\text{for}\quad |f|\le 1/(4T)\\ &\text{and}\quad H(f)=0\\ &\qquad\text{for}\quad |f|>1/(4T)\end{aligned} \tag{20}$$

for carrying out its filtering. The Fourier transform of the signal $x(t)$, that is $\mathrm{F}\,\big(x(t)\big)$, can be easily calculated as, for example, in (8). Then, we arrive at

$$\begin{aligned}&\mathrm{F}\,\big(x(t)\big)=\sum_{k=-\infty}^{\infty}\bar{x}[k]\exp(-j2\pi fkT)\\ &\quad\text{for}\ |f|\le 1/(2T)\\ &\text{and}\\ &\mathrm{F}\,\big(x(t)\big)=0\\ &\quad\text{for}\quad |f|>1/(2T)\end{aligned} \tag{21}$$

with $\bar{x}[k]$ denoting $\bar{x}[k]=T\cdot x[k]$. So, using (20) and (21), we can express the filtering mentioned above, in the frequency domain, as

$$
\begin{aligned}
&H(f)\cdot \mathrm{F}\left(x(t)\right)=\sum_{k=-\infty}^{\infty}\bar{x}[k]\exp(-j2\pi fkT)\\
&\quad \text{for } |f|\le 1/(4T)\\
&\quad \text{and}\\
&H(f)\cdot \mathrm{F}\left(x(t)\right)=0\\
&\quad \text{for } |f|> 1/(4T)\ .
\end{aligned}
\tag{22}
$$

In the next step, let us group all the components under the sum in the first line of (22) which contain the indices *k* that are even integers, and separate this group from the one which gathers all the components having *k* indices being odd integers. This will allow us to rewrite the first line of (22) in the following form:

$$
\begin{aligned}
&H(f)\cdot \mathrm{F}\left(x(t)\right)=T\sum_{k_e=-\infty}^{\infty}x[k_e]\exp(-j2\pi fk_eT)+\\
&+T\sum_{k_o=-\infty}^{\infty}x[k_o]\exp(-j2\pi fk_oT)=\\
&=T\sum_{n=-\infty}^{\infty}x_e[n]\exp(-j2\pi f2nT)+\\
&+T\sum_{n=-\infty}^{\infty}x_o[n]\exp(-j2\pi f(2n+1)T)=\\
&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\bar{x}_e[n]\exp(-j2\pi fn2T)+\\
&+\frac{1}{2}\sum_{n=-\infty}^{\infty}\bar{x}_o[n]\exp(-j2\pi fn2T)\exp(-j2\pi fT)
\end{aligned}
\tag{23}
$$

for $|f|\le 1/(4T)$. And, connecting (23) with the second line of (22), we arrive finally at

$$
\begin{aligned}
&H(f)\cdot \mathrm{F}\left(x(t)\right)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\bar{x}_e[n]\exp(-j2\pi fn2T)+\\
&+\frac{1}{2}\exp(-j2\pi fT)\sum_{n=-\infty}^{\infty}\bar{x}_o[n]\exp(-j2\pi fn2T)\\
&\quad \text{for } |f|\le 1/(4T)\\
&\quad \text{and}\\
&H(f)\cdot \mathrm{F}\left(x(t)\right)=0\\
&\quad \text{for } |f|> 1/(4T)\ .
\end{aligned}
\tag{24}
$$

Comparing now (24) with (19), we see that evidently the following:

$$
H(f)\cdot \mathrm{F}\left(x(t)\right)\neq \mathrm{F}\left(y(t)\right)\quad \text{for } |f|\le 1/(4T)
\tag{25}
$$

holds. That is the signal under-sampling operation is not the usual kind of low-pass filtering. It is a specific kind of signal shaping, involving also filtering. Its mechanism and effects are precisely explained above.

By the way, note that an interesting result, which regards our example studied throughout this paper, can be easily deduced from (24) and the remarks regarding the signal $z(t)$ introduced above. This result relates the outcome of filtering of the signal $x(t)$ with the use of the low-pass filter having the characteristic given by (20) and the outcomes of two under-samplings of this signal represented by the signals $y(t)$ and $z(t)$. It has the following form:

$$
\begin{aligned}
&H(f)\cdot \mathrm{F}\left(x(t)\right)=\\
&=\frac{1}{2}\left[\mathrm{F}\left(y(t)\right)+\exp(-j2\pi fT)\,\mathrm{F}\left(z(t)\right)\right]
\end{aligned}
\tag{26}
$$

and holds for all frequencies f . Finally, note that using the inversed Fourier transforms in (26), we get

$$
x_H(t)=\frac{1}{2}\left[y(t)+z(t-T)\right]\ ,
\tag{27}
$$

where $x_H(t)$ denotes the signal $x(t)$ filtered with the use of an ideal low-pass filter having the transfer function given by (20).

V. Concluding Remarks

It has been shown in this paper that the signal sampling operation can be considered as a kind of all-pass filtering in the time domain in the case when the Nyquist frequency is larger or equal to the maximal frequency in the spectrum of a signal sampled.

Obviously, if the Nyquist frequency is smaller from the maximal frequency in the spectrum of a signal sampled, then filtering occurs, too. However, it assumes then a specific form, which we have called here a signal shaping.

In this paper, we have proposed and illustrated a scheme for explanation of the signal shaping effects that occur in the case of signal under-sampling. Obviously, this scheme can be put into a more general framework.

References

[1] J. H. McClellan, R. Schafer, and M. Yoder, *DSP First*. London, England: Pearson, 2015.

[2] Ch. A. Bouman, *Digital Image Processing I - Lecture 11 - DTFT, DSFT, Sampling, and Reconstruction*. https://cosmolearning.video-lectures/dtft-dsft-sampling-reconstruction/, accessed December 2019.

[3] H. C. So, *Signals and Systems - Lecture 6 - Discrete-Time Fourier Transform*. www.ee.cityu.edu.hk/~hcso/ee3210.html, accessed December 2019.

[4] M. Vetterli, J. Kovacevic, and V. K. Goyal, *Foundations of Signal Processing*. Cambridge, England: Cambridge University Press, 2014.

[5] R. Wang, *Introduction to Orthogonal Transforms with Applications in Data Processing and Analysis*. Cambridge, England: Cambridge University Press, 2010.

[6] V. K. Ingle and J. G. Proakis, *Digital Signal Processing Using Matlab*. Stamford, CT, USA: Cengage Learning, 2012.

[7] W. K. Jenkins, *Fourier Methods for Signal Analysis and Processing* in W. K. Chen, *Fundamentals of Circuits and Filters*. Boca Raton, FL, USA: CRC Press, 2009.

[8] A. Borys, "Some topological aspects of sampling theorem and reconstruction formula," *Intl Journal of Electronics and Telecommunications*, accepted for publication in 2020.

[9] A. Borys, "Some useful results related with sampling theorem and reconstruction formula," *Intl Journal of Electronics and Telecommunications*, vol. 65, no. 3, pp. 471-475, 2019.