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The Problem of Aliasing and Folding Effects in Spectrum of Sampled Signals in View of Information Theory

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***Abstract*—In this paper, the problem of aliasing and folding effects in spectrum of sampled signals in view of Information Theory is discussed. To this end, the information content of deterministic continuous time signals, which are continuous functions, is formulated first. Then, this notion is extended to the sampled versions of these signals. In connection with it, new signal objects that are partly functions but partly not are introduced. It is shown that they allow to interpret correctly what the Whittaker–Shannon reconstruction formula in fact does. With help of this tool, the spectrum of the sampled signal is correctly calculated. The result achieved demonstrates that no aliasing and folding effects occur in the latter. Finally, it is shown that a Banach–Tarski-like paradox can be observed on the occasion of signal sampling.**

***Keywords*—signal sampling; modeling of sampled signal in the time domain; signal information content; spectrum aliasing and folding; Banach–Tarski-like paradox in signal sampling**

I. Introduction

THERE is a highly celebrated and commonly used (see, for example, [1]–[3]) expression for describing the spectrum of a sampled signal. It has the following form:

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - k/T), \tag{1}$$

where $X(f)$ means the spectrum of an energy signal $x(t)$ and $X(f-k/T)$ is this spectrum shifted by k/T (to the left or to the right, depending upon a sign of the integer k) on the frequency f axis. Moreover, t in $x(t)$ stands for a continuous time variable and $k = \dots,-1,0,1,\dots$. Further, T is a sampling period (in what follows, we use also its reciprocal $f_s = 1/T$, which means the sampling frequency (rate) used in sampling the signal $x(t)$.

Finally, if we denote by $x_s(t)$ the signal $x(t)$ sampled by "picking up" ideally its values periodically with the period T, then $X_s(f)$ occurring in (1) will mean its spectrum.

However, note that a similar formula as the one given by (1) is also valid when the sampling is not performed ideally. That is when the values of the signal "picked up" differ from those following from the function $x(t)$. Then, the non-ideal signal samples refer obviously to a new signal, which differs (in practice, slightly) from the (original) signal $x(t)$.

How to model this is explained shortly in what follows. So, to this end, let us denote the aforementioned signal as $x_{ni}(t)$; it will be associated with $x(t)$ in the following way:

$$x_{ni}(t) = x(t) + e_{av}(t), \tag{2}$$

where $e_{av}(t)$ means an error signal that follows from an effect of performing a local signal averaging operation before releasing a sample of $x(t)$. (The latter operation has been described in more detail in [4].)

As said, this model (developed in [4]) describes very well the operation of a non-ideal analog/digital conversion. In accordance with it, the samples of $x_{ni}(t)$, i.e. $x_{ni}(kT)$ are the ideal samples $x(kT)$ which are modified in such a way that the following: $x_{ni}(kT) = x(kT) + e_{av}(kT)$ holds. That is $e_{av}(kT)$ is an additive error to $x(kT)$. Furthermore, it follows from the latter equality that its form corresponds with the form given by (2). In other words, we can view $x(kT)$ and $e_{av}(kT)$ as representing "virtual" samples of (not available) signals $x(t)$ and $e_{av}(t)$, respectively.

It follows from (2) that the spectra $X_{ni}(f)$, $X(f)$, and $E_{av}(f)$ of the signals $x_{ni}(t)$, $x(t)$, and $e_{av}(t)$, respectively, are related with each other by

$$X_{ni}(f) = X(f) + E_{av}(f). \tag{3}$$

So, substituting $X_{ni}(f)$ given by (3) in (1) in place of $X(f)$

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there, gives

$$X_{sni}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_{ni}(f - k/T) = \\ = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - k/T) + \frac{1}{T}\sum_{k=-\infty}^{\infty} E_{av}(f - k/T), \quad (4)$$

where $X_{sni}(f)$ means the modified spectrum $X_s(f)$ – as a result of a non-ideal sampling operation performed by a (non-ideal) A/D (analog/digital) sampler.

The formula given by (1) or (4) (precisely, the expression on the right-hand side of (1) or (4)) shows evidently the occurrence of aliasing and folding effects in the spectrum of the signal $x_s(t)$ or $x_{sni}(t)$, respectively, where the latter means a sampled version of the signal $x_{ni}(t)$.

In this paper, we present a critique of the model that yields this result; it is done from the point of view of the Information Theory. Whereby, we limit ourselves here to consideration of the case of the ideal sampling only. It seems to us that the case of the non-ideal sampling will yield similar results, however, its detailed analysis is left for a later time or to others (if wished). A route is already outlined and framework prepared in this paper to perform the above task.

The remainder of the paper is organized as follows. In the next section, we define and discuss an information content of the deterministic continuous time signals which are continuous functions. Section III is a continuation of this subject. It is devoted to finding an appropriate definition of the information content of the deterministic continuous time signals which are sampled versions of continuous functions. The definition developed fully coincides with the one presented by Prof. Bracewell in [3]. However, their mappings in the time differ from each other substantially. As a result, their representations (spectra) in the frequency domain differ appropriately, which calls into question the validity of the formula (1). The next section is devoted to a certain paradox, called here a Banach–Tarski-like paradox, that can be observed when performing the signal sampling. It is explained here in detail. The paper ends with a final conclusion.

II. Information Content of Deterministic Continuous Time Signals which are Continuous Functions

Obviously, a question about an information content of the deterministic continuous time signals seems to belong to those peculiar or quirky ones – for various reasons.

First of all, it has nothing to do with the notion of entropy – despite that it asks about the information. This is so because we ask here about it in a quite different sense. Here, we speak about informational objects, which are not treated as probabilistic ones but are considered as deterministic functions.

Further, our objective here is not to find any quantitative measure for the information content contained in a given deterministic signal – in an usual sense of the word "measure" which is used in mathematics.

Instead of this, it is sufficient for us here to distinguish between different signals (functions), which carry some information as, for example, a voltage waveform registered in an electric (electronic) device. Such a signal carries information about values of the voltage at the device considered. It informs us about how they change with time. And, this is unique, distinct from other possible voltage waveforms.

So, for the needs of considerations presented in this paper, we assume any of the distinct waveforms, taken from the whole set of possible ones, as representing the information contained in it. In other words, any such waveform is also – per the above descriptive definition – its information content. Or, in our understanding, a signal and its information content mean the same.

Moreover, note also that when we speak in this section about signals of a continuous time (functions of a continuous time variable), we assume that they are, exclusively, continuous signals (functions).

The above assumption allows us to transfer the signals (functions) mentioned from the time domain into the frequency domain with the use of the standard Fourier transform or Fourier series. Furthermore, these transformations are one-to-one mappings. Therefore, we can say that the information contents of the corresponding spectra, obtained via mapping (time) signals with the use of the Fourier transform or Fourier series, retain the same.

In what sense? Referring to the field of topology, we could say that in a "topological" sense. That is they are converted into some other forms (similarly as geometric shapes), but without losing the ability to return to their original forms (that is functions of time).

In summary of this section, note that the information content of a continuous time signal can be equivalently expressed via its spectrum. Further, the latter (being a complex function of frequency) can be back-transformed, thereby allowing a perfect recovery to its original form.

III. Information Content of Deterministic Continuous Time Signals which are Sampled Versions of Continuous Functions

Consider the case of an ideal sampling of a continuous time signal being a continuous function that was considered in the previous section. So, if we denote it as $x(t)$, similarly as in the Introduction, with t meaning a continuous time variable (as before), its samples in this case will be exactly equal to its values picked up periodically with a period T.

Usually, in signal processing, one forms a sequence of indexed values from these samples. But, note that such a sequence cannot be treated as something like a "bag of indexed elements" only. Specifically, when one wants to calculate its spectrum in the sense which is inherently related with the notion of the standard Fourier transform or of the standard Fourier series. Then, we must take into account the fact that the indices of the sequence considered (understood here more generally as $kT \in \mathrm{R}, k \in \mathrm{Z}$, where R and Z are the sets of reals and integers, respectively) are in fact "deeply immersed" in the continuous timeline. In other words, the elements of the sequence are placed at strictly defined points of the continuous time space. So, it can be assumed then that they create a function of a continuous time. And, just in such a way, this is done in the signal processing literature.

More precisely, there exist in the literature (see, for example, [1] Fig. 4.2 (b) page 142; [2] Fig. 3.2 (c) page 36; [3] Fig. 10.2 (c) and (d) page 221) basically two ways of representing the indexed sequences $\{x(kT)\}=\{x_{kT}\}$, $k \in Z$ of samples of the signal $x(t)$ as a function (or a generalized function) of a continuous time. They are illustrated in Fig. 1.

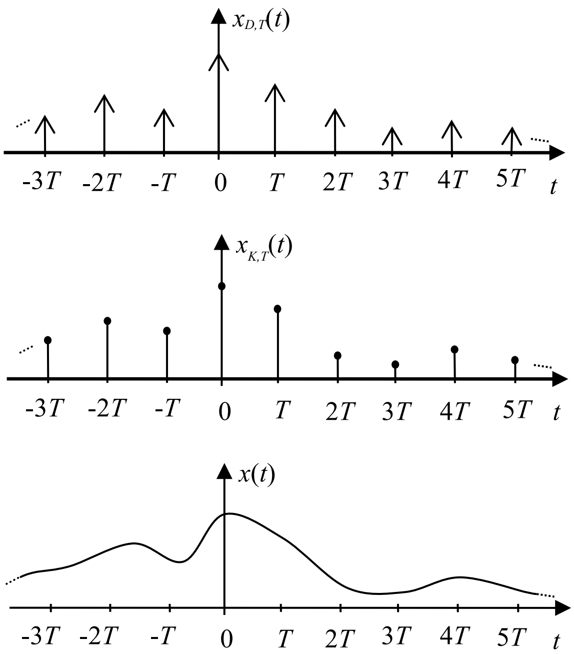

Fig. 1. Illustration to two graphical representations of a sequence of samples on the continuous time axis t which are used in the literature. They are a generalized function (upper curve) or a not continuous function (middle curve), respectively. In more detail, both of them are representations – associated with the sequence of samples – of an analog (i.e. un-sampled) signal shown at the bottom of this figure. Furthermore, note that the upper curve is a representation in form of a series of weighted Dirac deltas (generalized functions of this type) occurring uniformly on the continuous time axis t in the distance of T from each other. Whereas the middle curve is a series consisting of time-dependent signal elements (occurring also uniformly on the continuous time axis t in the distance of T from each other). Finally, we remark here that this figure is based on a one, which was used in discussions presented in [5] and [6].

The first of them uses a series of weighted Dirac deltas (i.e. generalized functions of this type) to model the sampled signal $x_s(t)$. That is this signal is expressed then as

$$x_s(t)=x_{D,T}(t)=\delta_T(t)\cdot x(t) \ , \tag{5}$$

where the so-called Dirac comb $\delta_T(t)$ is defined by

$$\delta_T(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT) \tag{6}$$

with $\delta(t-kT)$, $k=.,-1,0,1,.$, meaning the time-shifted Dirac deltas. Furthermore, note the use of an equivalent notation $x_{D,T}(t)$ in (5) and in Fig. 1 for $x_s(t)$. In it, the first index, D, stands for the name of Dirac. The latter notation (instead of $x_s(t)$) will be used in what follows to emphasize which model is currently in use (in modelling the sampled signal). Furthermore, we also emphasize here that the form of (1) follows exclusively from modelling of a sampled signal $x_s(t)$ by just a series of Dirac deltas (impulses). For more details regarding this fact see, for example, [1]–[3].

In the second way of the sampled signal modeling, which is illustrated in Fig. 1 (middle curve), we exclusively use the following notation: $x_{K,T}(t)$ – for denoting the sampled signal. In $x_{K,T}(t)$, the first index K stands for the name of Kronecker and the second one, T, means (as before) a sampling period. Furthermore, this signal can be expressed analytically as

$$x_{K,T}(t)=\delta_{K,T}(t)\cdot x(t) \ , \tag{7}$$

where $\delta_{K,T}(t)$ means a function that can be called, in analogy to the Dirac comb, a Kronecker comb [5]. And, the latter is then defined as

$$\delta_{K,T}(t)=\sum_{k=-\infty}^{\infty}\delta_{0,t/T-k}(t-kT)=\sum_{k=-\infty}^{\infty}\delta_{k,t/T}(t), \tag{8}$$

where $\delta_{i,t/T}(t)$, $i \in \mathbb{Z}$, denotes an extension of the usual Kronecker symbol to a function of time (as it has been done in [5]; more details on this can be found there).

It is worth noting at this point that the form of both the expressions on the right-hand sides of (5) and (7) is the same. That is it is a multiplication of the corresponding comb by the signal $x(t)$. Note however that these expressions are not identical because the combs involved in them are not identical. More precisely, we have $\delta(t-iT)\neq\delta_{i,t/T}(t)$, $i \in \mathbb{Z}$, what implies $\delta_T(t)\neq\delta_{K,T}(t)$, and finally $x_{D,T}(t)\neq x_{K,T}(t)$.

Note that contrary to the above Prof. Bracewell writes in a caption to Fig. 10.2 on page 221 of his excellent book [3] the following: "The samples (c) [that is those which are shown by a curve (c) of his Fig. 10.2, which in turn is visualized here by the middle curve of our Fig. 1] are equivalent in content to the train of impulses (d) [that is those which are shown by a curve (d) of his Fig. 10.2, which in turn is visualized here by the upper curve of our Fig. 1]. Obviously, mathematically speaking, such a claim is not correct because real numbers cannot be simply considered (treated) as Dirac impulses and vice versa. However, see that the Prof. Bracewell's claim of "equivalence in content", expressed above, can be made understandable correctly in the sense of identification of the coefficients of the impulses (i.e. Dirac deltas multiplied by some numbers) in his Fig. 10.2 (d) with the corresponding numbers shown on the curve (c) of his Fig. 10.2. And, with assuming at the same time that all the rest have no meaning.

Let us further stick to the monograph of Prof. Bracewell [3] while discussing a not understandable, in our opinion, identification of the sampling process with the Dirac comb (in a description of this process) and/or saying that both the representations of the sampled signal mentioned above, i.e. $x_{D,T}(t)$ and $x_{K,T}(t)$, are identical (or that $x_{D,T}(t)$ is the only valid one). On page 221 of [3], Prof. Bracewell writes: "In the derivation that follows, the introduction of the *shah* symbol [named here the Dirac comb and denoted as $\delta_T(t)$] proves convenient, because multiplication by Ш(t) [that is by $\delta_T(t)$, using our notation] is equivalent to sampling, in the sense that information is retained at the sampling points and abandoned in between." A little bit further, on the next page, he continues: "Consider the function $\sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$ [expressed with the use of our notation; further, note that it is another form of (5), in which (6) and the sifting property of the Dirac delta have be applied] shown in Fig. 10.2 (c–>d). Information about $x(t)$ is conserved only at the sampling points where t is an integral multiple of the sampling interval T. The intermediate values of $x(t)$ are lost."

See now that the above sentences quoted confirm that [3] models the signal sampling as an operation performed by a Dirac comb on a signal $x(t)$ (more precisely, it is defined as an operation of their multiplication). But, it is surprising there that an another function, namely $\sum_{k=-\infty}^{\infty} x(kT)\delta_{0,t/T-k}(t-kT)$ (which is visualized here in Fig. 1 (middle curve)), is indicated by Prof. Bracewell as a result of this operation (i.e. of this multiplication). Not the function $\sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$ mentioned by him; the latter is visualized (here) in Fig. 1 (upper curve)). This suggests of course that the functions $\sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$ and $\sum_{k=-\infty}^{\infty} x(kT)\delta_{0,t/T-k}(t-kT)$ are used by Prof. Bracewell interchangeably, as equivalents of each other. However, this is mathematically forbidden because, as already mentioned, $\delta_T(t) \neq \delta_{K,T}(t)$, what follows from the fact that $\delta(t-kT) \neq \delta_{0,t/T-k}(t-kT)$, $k \in \mathbb{Z}$ (for more details, see [5]).

Furthermore, note that the reasoning of Prof. Bracewell presented above can be also interpreted as follows. He assumes that an internal behavior of an analog signal sampler (A/D converter) can be modelled, in a quite abstractive way, with the use of the Dirac comb. But, this Dirac comb does not "appear" in any form at the output of the A/D converter. Instead, the signal of the form presented in Fig. 1 (middle curve) appears at its output.

Obviously, the above interpretation casts doubts on the meaningfulness of that abstraction which exploits the Dirac comb.

However, we cannot end at this point our discussion of the possible representations of a sampled signal, as they are sketched in Fig. 1 (upper and middle curves). Why? Because a closer look at them, at their descriptions presented in [1]–[3], shows that they differ or can differ slightly from each other with regard to the sets of those instants of the t axis at which no signal sampling is performed. More precisely, Prof. Bracewell by saying "abandoned in between" (with regard to the object shown in Fig. 10.2 (d)) and that "the intermediate values of $x(t)$ are lost" (with regard to the object shown in Fig. 10.2 (c)) tells us in fact that the values of the objects mentioned for all the instants different from the sampling points are not known (or are not defined). And, relating this to our curves from Fig. 1, see that it would mean that all the values of $x_{D,T}(t)$ and $x_{K,T}(t)$ for the instants different from the sampling points would not be equal to zeros (as suggested there), but they would be simply undefined (or unknown).

Contrary to the above, we have in [1] and [2] a quite different interpretation. To see this, look at page 141 of [1], where Oppenheim, Schafer, and Buck write "$x_s(t)$ [which corresponds to our $x_{D,T}(t)$ in Fig. 1 (upper curve)] is, in a sense, a continuous-time signal (specifically, an impulse train) that is zero except at integer multiples of T". Whereby, note that this assumption (or fact) follows directly from a simplified (naive) definition of the Dirac delta which is used in [1] and [2]; for more details and discussion about the latter definition, see, for example, [7].

The possibility of modelling the sampled signal in form of a one that is presented in Fig. 1 (middle curve) has not been considered in [1] and [2]. However, we can presume that if this were done there, it would have been made similarly. That is all the values of this signal (i.e. of $x_{K,T}(t)$ in our notation) lying between the sampling instants would be identically equal to zeros.

Let us now summarize our above findings regarding modelling of the sampled signal. First of all, the main conclusion that follows from them is the following: there are in fact four proposals in the literature to model this signal. However, they are not identical – although people think it is so. In what follows, we will strive to disabuse the researchers of that delusion. To this end, we start with defining (formally) and denoting appropriately all the signal forms discussed above. So, from now on, we use the notation $x_{D,T}(t)$ exclusively for the signal described by the right-hand side of (5) and with all its values equal identically to zero for instants different from the sampling points. (Note that this understanding of $x_{D,T}(t)$ is based on the simplified (naive) definition of the Dirac delta – mentioned above.) Further, the notation $x_{K,T}(t)$ exclusively for the signal described by the right-hand side of (7) and with all its values equal identically to zero for instants different from the sampling points. (Note that no comment need to be added in this case because the right-hand side of (7) determines precisely the values of this signal.) Moreover, we introduce a new notation, $\tilde{x}_{D,T}(t)$, for the signal described by the right-hand side of (5) but now with all its values unknown (not defined) for instants different from the sampling points. (Note that this understanding of $\tilde{x}_{D,T}(t)$ can be attributed to a more sophisticated definition of the Dirac delta – considered as a distribution (generalized function)). And, finally, we introduce another new notation $\tilde{x}_{K,T}(t)$ for the signal described by the right-hand side of (7) but only for the instants of signal sampling and with all its values unknown (not defined) for the other instants (that is for those which are different from the sampling points). (Note that this signal can be viewed as a result of applying an additional operator, say *U*, to the signal $x_{K,T}(t)$, with assuming that this operator makes all the zero values of $x_{K,T}(t)$ outside the sampling points undefined ones. That is we have $\tilde{x}_{K,T}(t)=U\left(x_{K,T}(t)\right)$. Furthermore, observe that then the signal $\tilde{x}_{K,T}(t)$ does not represent a function (because it does not meet the function definition) – in contrast to $x_{K,T}(t)$, which is an ordinary function.)

It follows clearly from the above discussion that the signals $x_{D,T}(t)$, $x_{K,T}(t)$, $\tilde{x}_{D,T}(t)$, and $\tilde{x}_{K,T}(t)$, are not equivalent to each other. Nevertheless, all the four are used in modelling of the same thing: the output signal of A/D converters. So, this is rather a paradoxical situation. However, see that it would be reasonable and appropriate to ask which of them (in our opinion) most closely reflects the reality.

The signals $x_{D,T}(t)$ and $\tilde{x}_{D,T}(t)$ seem to be not good candidates for modelling correctly the reality, for a very simple reason. At the output of A/D converters, the Dirac deltas do not appear, as already mentioned. Instead of this, there appear finite values (numbers) at the output of A/D converters, in a “rhythm” determined by the sampling rate. And, see that this behavior is perfectly reflected in the signal $x_{K,T}(t)$. So, we must accept it (there is no other choice) as the one which reflects the reality most closely.

But what about the signal $\tilde{x}_{K,T}(t)$? It is very useful in signal processing because it points us, indirectly, to an appropriate signal of a continuous time, which was sampled, and which we will shortly want to recover – by performing the signal reconstruction along the lines of the Whittaker–Shannon formula [1]–[3]

$$x(t)=\sum_{k=-\infty}^{\infty}\tilde{x}_{K,T}(kT)\,\mathrm{sinc}(t/T-k)\ , \tag{9}$$

where the function $\mathrm{sinc}(t)$ is defined as

$$\mathrm{sinc}(t)=\sin(\pi t)/\pi t \quad \text{for } t\neq 0 \quad \text{and } 1 \text{ for } t=0\ . \tag{10}$$

Further, note also that the signal $\tilde{x}_{K,T}(t)$, which is not equal to the real signal $x_{K,T}(t)$ occurring at the output of the A/D converter, must be treated as a virtual signal. However, it is needed in signal processing.

To demonstrate its necessity, let us now carry out the following thought experiment (*Gedankenexperiment*) – as presented below. Thus, assume for a while that a signal $x(t)$ to be sampled has a form of the one illustrated in Fig. 1 (middle curve). That is all its values referring to the points outside the points $t=kT$, $k\in\mathrm{Z}$, on the time axis are identically equal to zero. Next, perform the sampling operation on this signal, choosing sampling instants exactly at the points $t=kT$, $k\in\mathrm{Z}$. In this way, we obtain a sampled signal $x_{K,T}(t)$ that will be identical with $x(t)$. As a consequence, the recovery of the latter signal with the use of the formula (9) would lead, in this case, to an obvious error. However, see that we would avoid this error, if we noted that the function $\tilde{x}_{K,T}(t)$ is also equal to $x_{K,T}(t)$ in this case. Why? Because simply in this case we do not need any signal recovery process. Or, in other words, all the “theoretically” undefined values after performing sampling are here known.

In turn, assume now $x(t)$ to be a bandlimited signal $x(t)$ we want to sample. Perform then the sampling operation on this signal, choosing the sampling instants exactly at the same points $t=kT$, $k\in\mathrm{Z}$ as before. And, assume further that all the samples of this signal are exactly the same as before. That is the function $x_{K,T}(t)$ is the same in both cases. However now, unlike in the previous case, the function $\tilde{x}_{K,T}(t)$ differs from $x_{K,T}(t)$. And, this fact indicates that new values will appear in the recovery process (here according to the rule given by (9), or, in general, to any other appropriate one) at the instants at which zeros (per definition) occur in $x_{K,T}(t)$.

So, in summary of the above, we can say that if the following: $\tilde{x}_{K,T}(t)\neq x_{K,T}(t)$ holds, this means that a non-pathological signal has been sampled. And, a signal reconstruction reveals new signal values in this case.

Additionally, note that before performing the signal reconstruction a kind of digital filtering on the signal samples is often performed. Obviously, it is carried out with the use of a signal processor (which works on numbers); further, these manipulations on numbers cause that the function $x_{K,T}(t)$ and the associated one, $\tilde{x}_{K,T}(t)$ are modified. So, the modified function $\tilde{x}_{K,T}(t)$ shows in this case all the “hidden” values of the signal which need to be “uncovered” and which were also (in between, before signal reconstruction) filtered. This ends our proof of the necessity of the function $\tilde{x}_{K,T}(t)$.

We take also this opportunity to bring to the reader’s attention the fact that any signal $x(t)$ being a continuous function of a time variable t, together with all its sampled versions $\tilde{x}_{K,T}(t)$ containing sets of "hidden" values that can be perfectly "uncovered", can be considered on a unified basis as a signal object [8], or as a function with an attribute [9]. For more details, see [8] and [9].

Further, we remark also that (9) can be rewritten as

$$x(t)=\sum_{k=-\infty}^{\infty} x_{K,T}(kT)\,\mathrm{sinc}(t/T-k) \tag{11}$$

because the values of the signals $x_{K,T}(t)$ and $\tilde{x}_{K,T}(t)$ are the same at the sampling points $kT,\ k\in \mathrm{Z}$. (Moreover, see that $x_{D,T}(kT)=x_{K,T}(kT)=x(kT),\ k\in \mathrm{Z}$, what follows from (5) and (7).)

Let us emphasize once again the importance of our finding that just the signal $\tilde{x}_{K,T}(t)$ is really the one which most closely reflects the reality at the A/D converter output at the sampling points and, at the same time, informs us which of the other values of this function (per definition) are not known, but will not be identically equal to zeros after performing the signal reconstruction. Furthermore, observe that the graph of $\tilde{x}_{K,T}(t)$ corresponds with the Prof. Bracewell's graph presented at Fig. 10.2 (c) on page 221 of [3]. And that he said (there) the following: "Information about $x(t)$ is conserved only at the sampling points where t is an integral multiple of the sampling interval T." – as we have already cited.

Further, note that this is the whole information contained in $x(t)$. What is this due to? This follows from the fact that $x(t)$ can be perfectly reconstructed just from the information mentioned via the formula (9). Whereby, when T does not satisfy the conditions of the Nyquist–Shannon sampling theorem [1]–[3], $x(t)$ is to be understood as a bandlimited modified signal (where, here, the signal modification means a kind of filtering connected with the signal shaping at the same time; they have been described as well as explained in detail in [10]).

Can we equivalently express this information in the frequency domain? In a similar way as it was done in the case of the deterministic continuous time signals being continuous functions, discussed in Section II? This is not possible. Why? Because $\tilde{x}_{K,T}(t)$ is not a function (as pointed out before). Therefore, its Fourier transform does not exist.

Note also that the other possible candidates mentioned before: $x_{K,T}(t)$ and $x_{D,T}(t)$ do have Fourier transforms, but they contain false information about the sampled signal information content. The former because its Fourier transform is equal to zero (for more details about, see [5]). And the latter because its spectrum, expressed by (1), is burdened by the effect of multiple duplication.

However, see that, if we so strongly insist on characterizing the information content of the sampled signal also by a spectrum, it follows from all the considerations presented above that the only reasonable solution is then to assign to it the spectrum of $x(t)$, i.e. its Fourier transform $X(f)=\mathrm{F}\,(x(t))$, where $\mathrm{F}\,(\cdot)$ stands for carrying out the standard Fourier transformation. Moreover, note that the proposal and arguments for doing so have been already presented in the literature, see [5] and [6]. We continue here this topic.

Whereas, the above must be slightly modified when the sampling period T does not satisfy the conditions of the Nyquist–Shannon sampling theorem [1]–[3]. More precisely, when the following:

$$T\le 1/(2f_m) \tag{12}$$

does not hold, where f_m stands for the maximal frequency in the spectrum of the bandlimited signal $x(t)$. Then, we must take a modified signal $x_a(t)$, calculated from

$$x_a(t)=\sum_{k=-\infty}^{\infty}\left(x_a(kT)=\tilde{x}_{K,T}(kT)\right)\mathrm{sinc}(t/T-k), \tag{13}$$

instead of $x(t)$, in calculation of the corresponding spectrum via the relation $X_a(f)=\mathrm{F}\,(x_a(t))$. (Whereas, it is still worth to remember that $\tilde{x}_{K,T}(kT)=x_{K,T}(kT)=x(kT)$ is all the time in force.)

The signal $x_a(t)$ given by (13) – that is when (12) does not hold – differs obviously from $x(t)$. (As a result, we have $X_a(f)\neq X(f)$, too.) We can say that the former signal is distorted when comparing it with the latter one. However, it still remains a bandlimited one. More precisely, as shown in [10], it can be viewed as a result of low-pass filtering of the signal $x(t)$ and its shaping at the same time. And, the resulting signal possesses the maximal frequency that is present in its spectrum – denote it here as f_{ma} – equal to $1/(2T)$ (that is $f_{ma}=1/(2T)$).

So, concluding the above outcomes, we see that the information content of the sampled signal, considered equivalently in the frequency domain, is given by

$$\mathrm{SPECT}\left(\tilde{x}_{K,T}(t)\right)=\begin{cases}X(f) & \text{when } T\le 1/(2f_m)\\ X_a(f) & \text{when } T>1/(2f_m)\end{cases}, \tag{14}$$

where a new symbol $\mathrm{SPECT}(\cdot)$ is used (because the Fourier transform of $\tilde{x}_{K,T}(t)$ does not exist).

Observe further that the right-hand side of (14) differs substantially from the one in (1). Here, no aliasing and folding effects occur. And, in our opinion, it represents a correct model.

Let us now describe the process of signal sampling via the language and notions of Information Theory. And, to this end, observe that the signal sampling can be viewed as a process of data (information) compression, in which the signal $x(t)$ can be seen as possessing much redundant information. This redundant information is removed from it through the sampling operation, and as a result we obtain the signal $\tilde{x}_{K,T}(t)$. If now the sampling period T satisfies the inequality (12), all the redundant information (indicated in $\tilde{x}_{K,T}(t)$ as undefined values) can be fully recovered from $\tilde{x}_{K,T}(t)$ via application of (9). Therefore, we can call this sampling a lossless one; it corresponds with the lossless compression of information. Unlike this, when the sampling period T does not satisfy (12), not only a redundant information is removed, but also some amount of the non-redundant one, too. In other words, some information is then lost and as a consequence, the signal $\tilde{x}_{K,T}(t)$ does not enable to recover the signal $x(t)$ (only the signal $x_a(t)$ that is a distorted signal – with regard to $x(t)$). So, we can name this sampling a lossy one; it corresponds with the lossy compression of information.

Note that taking the above into account we can say equivalently that the information content of the sampled signal (its spectrum) equal to $X(f)$ means the lossless sampling, but equal to $X_a(f)$ the lossy one.

IV. Banach–Tarski-Like Paradox in Signal Sampling Process

This section is devoted to a certain effect, which can be observed in the signal sampling process, namely a multiplication of the signal that is sampled. In other words, we can observe here an effect which is similar to the one occurring in the so-called Banach–Tarski paradox [11]. The latter deals with such a decomposition of the three-dimensional ball that allows to build up two copies of it from the pieces obtained in this decomposition. As we show here, the signal sampling can be also viewed as a partitioning process. Moreover, it has specific properties.

Let us start our considerations here with defining a new signal $x^c_{K,T}(t)$ that is complementary to $x_{K,T}(t)$. That is with

$$x^c_{K,T}(t) = x(t) - x_{K,T}(t). \tag{15}$$

From this definition, it follows that the signal $x^c_{K,T}(t)$ represents the values of the signal $x(t)$ for all the instants except of those which are the sampling points. At the latter points, the values of the signal $x^c_{K,T}(t)$ – considered as a function – are equal to zero. Complementary to that, the signal $x_{K,T}(t)$ contains the values of samples of the signal $x(t)$ (placed obviously at the corresponding instants). But all the other values of the former signal – considered as a function – are identically equal to zero.

In the next step, let us replace "all these holes", that is the places in the functions $x_{K,T}(t)$ and $x^c_{K,T}(t)$ in which we have the aforementioned zeros, with unknowns. In other words, let us treat, in what follows, these zeros as some values which are not known. And, see that this in fact leads to building up new objects, $\tilde{x}^c_{K,T}(t)$ and $\tilde{x}_{K,T}(t)$ on the basis of the functions $x^c_{K,T}(t)$ and $x_{K,T}(t)$. (The former ones are not functions. Moreover, note that $\tilde{x}_{K,T}(t)$ has been already defined in Section III, and $\tilde{x}^c_{K,T}(t)$ is defined in a similar way.)

Let us assume now that the sampling period T is so chosen here that (12) holds. Then, as the Nyquist–Shannon sampling theorem [1]–[3] states and in connection with its application to the signal $\tilde{x}_{K,T}(t)$ as discussed in Section III, we can get a perfect image of the signal $x(t)$ from it. And, we do that at this moment, getting the first copy of $x(t)$. In other words, we get $\tilde{x}_{K,T}(t)$ with all its unknowns replaced by the "true" values of $x(t)$ at the corresponding points (which, as the whole, provides us with $x(t)$).

Obviously, we will get the second copy of $x(t)$ by manipulating on the signal $\tilde{x}^c_{K,T}(t)$. Here, the manipulation will rely upon finding the limits of $\tilde{x}^c_{K,T}(t)$ for the instants at which its unknowns occur and replacing the latter ones just by these limits found. In more detail, we calculate the left-hand side limit, $\lim_{kT_- \to kT}\left(\tilde{x}^c_{K,T}(kT_-)\right)$, $k \in \mathbf{Z}$, or the right-hand side one, $\lim_{kT_+ \to kT}\left(\tilde{x}^c_{K,T}(kT_+)\right)$, $k \in \mathbf{Z}$. Further, see that the continuity property of the function $x(t)$ ensures that the above limits exist and are equal to each other. We pick up one of them and put it into the place where the corresponding unknown in the signal $\tilde{x}^c_{K,T}(t)$ occurs. After performing this at all the instants kT, $k \in \mathbf{Z}$ (countably many), we get from $\tilde{x}^c_{K,T}(t)$ the signal $x(t)$. And, this is the second copy of it received in our partitioning process. Hence, really, we have duplicated the signal $x(t)$.

Consider here also the case of choosing the sampling period T so that it does not satisfy the condition (12). Then, we obtain obviously the "compressed" signal $x_a(t)$ after applying the Whittaker–Shannon formula (9) to the samples of $\tilde{x}_{K,T}(t)$ in the first part of the procedure described above. But, in its second part, we obtain the signal $x(t)$ from $\tilde{x}^c_{K,T}(t)$ – as before (worth emphasizing). So, in this case, the signal duplication procedure will have two branches: the one that is "non-perfect" and the second being fully perfect one.

Finally, let us note that the procedure described above can be repeated an infinite number of times generating an infinite number (but countable) of signals $x(t)$ (when (12) is satisfied) or pairs $x_a(t)$ and $x(t)$ (when (12) is not satisfied).

V. Final Conclusion

The results obtained in this paper show that by using simple mathematical tools and basic notions of Information Theory it is possible to resolve the problem of occurrence of the aliasing and folding effects in spectrum of sampled signals. The outcomes of this paper indicate that such effects do not occur.

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