High – gain and wide – bandwidth patch antenna for fifth generation communication applications at K - Band

Safa N. Nafea^{*}, Nasser N. Khamiss

Abstract—The patch antenna usually used for many wireless applications, and fifth generation (5G) is the most attracting application in the field of millimeter wave communications recently. The researchers worked on solving problems those considered milestones against using patch antenna for 5G applications such as being an antenna with moderate gain and narrow operating bandwidth with high side lobe levels. In this article a microstrip patch antenna had been presented to operate over the n258 – Band for 5G communications at 26 GHz with an operating bandwidth around than 7 GHz. The proposed antenna was printed on Rogers RT/Duroid 5880 substrate. A high gain of 8.10 dB had been achieved with high Front - to - Back ratio of 24.47 dB and very low side lobe levels far field radiation pattern around -17 dB. The proposed antenna covered the operating bandwidth n258 for fifth generation applications in range of (24.25 - 27.50) GHz. The Computer simulation Technology (CST) had been used as a simulation environment for this design.

Keywords—Gain; Microstrip Patch Antenna; Millimeter Wave Communications; Operating Bandwidth; Side Lobe Levels; 5G

I. INTRODUCTION

HE patch antenna shows many advantages such as being a low cost and weight antenna with ability of operating for more than a single resonance frequency at the same time as well as being an antenna with ease fabrication process [1]. This type of antenna suffers from being a low gain antenna with narrowband and high side lobes level (SLL) radiation pattern [1]. Many known techniques have been used to improve gain of patch antenna such as using an array of patches with circular shape which had diameter of 325 mm to achieve high gain of 29 dB [2]. In [3] 2×1 patch array covered with a superstrate of metamaterial which caused gain increment from 4.30 dB to 8.00 dB. The use of multilayer patch antenna caused an enhancement in general performance of antenna especially gain of antenna. Therefore covering a single patch antenna with three layers of Rogers RO3006 caused to improve the gain about 3.5 dB [1]. While adding 5×5 partially reflecting surfaces on bottom of single FR4 superstrate; located above feeding patch, increased gain of antenna around 5.5 dB [4].

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Moreover the shape of antenna's ground has an effect on its gain as in case of using a grooved ground plane with ten grooves or more per wavelength which caused an increment of 10 dB [5]. Moreover using defected ground with a reflecting layer located under antenna's ground caused gain improvement [6]. Using an etched ground plan caused to have triple operating frequencies for WiMax and WLAN application at 3.3 GHz , 4.5 GHz , and 5.8 GHz with gain ranging from 4.33 dB to 4.85 dB [7]. As well as using an array of etched slots from radiating surface of patch causes an increment in antenna's gain around 13 dB [8]. Using 2x1 of patch array located above a layer of 2x4 of artificial magnetic conductors (AMC) caused to achieve gain of 7 dB [9]. Gain increment around 2 dB was achieved using 7x7 array of near – zero indexed metamaterial [10]

Researches had been conducted and utilized many techniques to improve the operating bandwidth of patch antenna such as using multi-layer antenna structure which shows an effective performance improvement by increasing gain and bandwidth of antenna with obvious reduction in SLL in radiation pattern and reflection coefficient [1]. Moreover adding metamaterial on top of superstrate material would improve the operating bandwidth of antenna as in case of split resonators metamaterial [3]. Adding Rogers RO3010 superstrate above a single layer patch caused an increment in antenna bandwidth [11]. Using an array of etched square slots is an effective technique to improve bandwidth of antenna [8]. The use of defected ground rather than regular or traditional ground cased to have a wide operating band of 5.13 GHz and 11.63 GHz for 5G antenna operating at both 28 and 38 GHz, respectively [12]. In [13] a mathematical model had been developed to study the effect of patch's conductor thickness and material on general performance of the antenna, which would lead to achieve wider operating bandwidth. The use of specific shaped slots in patch of antenna would improve the antenna's -3dB bandwidth, where the slotted patch achieved an operating bandwidth around 1.1 GHz at 28 GHz [14]. Another slot shaped Pi patch caused to achieved wider operating band range more than 7 GHz [15].

The patch antenna had been used for 5G applications within many bands: S - Band, C - Band, K - Band, and Ka - Band. A reconfigurable antenna had been proposed in [16] to operate at 3.6 GHz with gain of 4.8 dB, while [17] presented an antenna

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resonates at 3.7 GHz with gain of 4.3 dB. As we mentioned earlier in this section, patch antenna can operate for multiple operating frequencies which had been demonstrated in case of multi-band patch antenna for 5G applications at S and C – Band with gain ranging from 5.5 dB to 8.2 dB [18], as well as patch antenna poposed by [14], was designed to operate for 5G applications at 26 GHz and 28 GHz.

Patch antenna can be also used for 5G applications at Ka -Band where slot was ethched from line feeding of patch antenna designed to 5G at 28 Ghz caused gain of 2.6 dB [19]. Moreover an antenna designed to operate at 28 GHz printed on Rogers RT / Duroid 5880 with gain of 2.6 dB with return loss of -34 dB [20]. While the proposed antenna by [21] operated at 38 GHz and achieved high gain of 12 dB. An 2x1 MIMO patch antenna was deigned to operate for both K and Ka – Bands at operating frequencies of 28 GHz and 38 GHz for 5G communications [22]. A hig gain antenna with two flipped T – shaped slots in patch surface was introduced to operate at 28 GHz and 38 GHz [23]. In [24] a patch antenna was introduced to operate with four frequency bands K, Ka, V, and W; the operating frequencies of antenna are 23.8 GHz, 39.4 GHz, 66.2 GHz, 81.9 GHz and 93.9 GHz, where 23.8 GHz and 39.4 GHz could be used for 5G communications in k and Ka- Bands.

The fifth generation wireless communication is high bit rate antenna which requires a high gain antenna with wider operating bandwidth, therefore [13] developed a mathematical model to study effects of copper thickness on antenna's gain at 28 GHz, while the proposed antenna in [25] used ground plane of 8x8 array of AMC to enhance antenna's gain at 28.4 GHz.

To achieve wider operating bandwidth a defected ground structure havs been used to improve antenna's bandwidth [12], while patch antenna in [15] used pi – shaped slots to increase the bandwith over Ka- Band. Using a partial ground plane also improved antenna's bandwidth for the porposed antenna by [26]. Moreove a reduced size patch antenna with an operating bandwidth of 1.4 GHz was proposed to operate in Ka- Band [27].

In this article a single layer patch antenna with low SLL and high gain at 26 GHz which covers the n258 band for 5G applications (24.25 - 27.5) GHz was presented. The proposed antenna has very low SLL radiation pattern without any need for any techniques such us using multi layer antenna structures in [1] or using reflector layer as in case of [6] The proposed single – element patch antenna shows higher gain compared to antenna proposed in [19] compared with lower physical area. The proposed single - element antenna achieved a comparable gain with reduced physical area compared to two elements MIMO antenna proposed in [22]. Finally a higher gain with lower physical area was achieved by the proposed antenna compared to low profile antenna by [24]. The proposed antenna achieved an operating bandwith wider than achieved by LI slotted antenna proposed by [14] with a comparable gain. The 5G communications supports multiple frequency bands: below 2GHz, above 2GHz, and (24 – 52.6) GHz, for many applications such as UHD video streaming, smart cities, wireless sensing, and smart home [27-28].

II. ANTENNA DESIGN

The proposed patch antenna for 5G communications application was etched on low loss Rogers RT/Duroid 5880 substrate with over all dimensions of (L×W×h); where L is the length of the substrate, W is the width of the substrate, and h is the thickness of the substrate. The substrate has dielectric constant (ε_r) of 2.2 and loss tangent (tan δ) of 0.0009.

The radiating patch is a rectangular shaped patch with three circles; one was located above the upper side of the rectangular patch and the distance between patch and central circle is (d_{CC}), while the other two circles are located above the right-hand and left-hand side corners of the rectangular patch and the distance between patch and both circles is (d_{SC}). The rectangular patch has dimensions of ($L_P \times W_P$), where L_p is the length and W_p is the width of the patch.

Each of the three circles has a circular shape was etched from the center of the circle with radius of (r_{IN}) . The outer radius for the central circle is (r_{CC}) , while the (r_{SC}) is the outer radius for both other two circles. Two right angled triangles with side length of (T_{SL}) had been used for connecting the central circle to both right-hand and left-hand circles.

The proposed patch antenna for 5G applications at K - Band had been fed using a microstrip line feeder which has length and width of (F_L) and (F_W), respectively. The copper thickness is (t). The ground plane of the antenna was fully covered by copper. Table I bellow presents a List of parameters and dimensions for the proposed 5G patch antenna design. Fig. 1 shows top view for the proposed antenna.

 TABLE I

 LIST OF PARAMETERS AND DIMENSIONS FOR THE PROPOSED DESIGN

No.	Parameter	Dimension		
1	Length of Substrate - L	Substrate - L 8.20 mm		
2	Width of Substrate - W 8.50 mm			
3	Thickness Of Substrate - h	1.57 mm		
4	Substrate's Dielectric Constant - $\varepsilon_{\rm r}$	2.20		
5	Substrate's Loss Tangent - tan δ 0.0009			
6	Length of Rectangular Patch - L _P 2.85 mm			
7	Width of Rectangular Patch - W_P	4.00 mm		
8	Distance Between Patch and Central Circle - 1.00 mm			
	d _{CC}			
9	Distance Between Patch and Both Right-	0.60 mm		
	Hand and Left-Hand - $d_{\rm SC}$			
10	$\label{eq:outer Radius for Central Circle - r_{CC} \qquad \qquad 0.40 \ \text{mm}$			
11	Outer Radius for Both Right-Hand and Left- 0.20 mm			
	Hand Circles - r _{SC}			
12	Inner Radius for All Circles 0.05 mm			
13	Right Angled Triangles' Side Length - T_{SL}	1.80 mm		
14	Length of Microstrip Line Feeder - F_{L}	2.50 mm		
15	Width of Microstrip Line Feeder – F_{W}	0.40 mm		

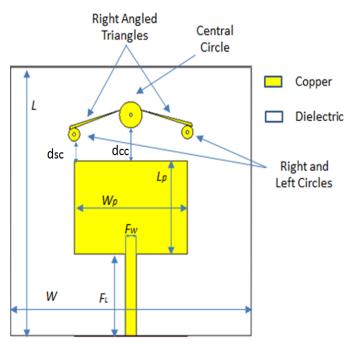


Fig. 1. Top View For The Proposed Patch Antenna For 5G Applications At K-Band

III. RESULTS

This section presents the parametric study which had been used in designing the proposed patch antenna to achieve the optimum design results in terms of reflection coefficient, gain, bandwidth and resonance frequency. This study can be summarized by following steps:

1. Designing a rectangular patch to operate at a resonance frequency of 26 GHz. The optimum dimensions for the patch, L_P and W_P were optimized to be 2.85 mm and 4.00 mm, respectively. In this step the frequency of antenna was 25.96 GHz with a reflection coefficient of -34.97 dB, while the gain of antenna was 8.09 dB. Table-II shown below presents steps for optimizing the dimensions of the rectangular patch.

TABLE II			
OPTIMIZATION STEPS FOR L _P AND W _P			

				1 1	
No.	$L_{P}\left(mm ight)$	$W_{P}(mm)$	Frequency (GHz)	S ₁₁ (dB)	VSWR
1	3.00	4.50	24.82	-44.50	1.012
2	2.97	4.40	25.08	-41.13	1.018
3	2.94	4.30	25.36	-38.53	1.024
4	2.91	4.20	25.66	-36.47	1.030
5	2.88	4.10	26.26	-33.39	1.044
6	2.85	4.00	25.96	-34.97	1.036

2. The second step in this study is adding the central circle with outer and inner radius of 0.40 mm and 0.05 mm,

respectively. Adding the circle caused an operating frequency of 26.09 GHz with obvious reduction in reflection coefficient to be -50.64 dB. The VSWR of 1.006 and achieved gain of antenna was 8.10 dB.

- 3. The third step includes adding right-hand and left-had side circles with outer and inner radius of 0.20 mm and 0.05 mm, respectively, for both of the circles. In this step the resonance frequency was 26.15 GHz with a reflection coefficient of -35.31 dB. A slight reduction in gain to be 8.00 dB was noticed, while VSWR found to be 1.035.
- 4. The final step in designing the proposed patch antenna for 5G communications at K Band, is adding two right angled triangles with side length of 1.80 mm which connecting central circle with both right-hand and left-hand circles. The operating frequency shifted to 26.08 GHz with obvious reduction in reflection coefficient to be -47.38 dB with VSWR of 1.008. The gain of antenna increased to be 8.10 dB compared to previous step with very low SLL radiation pattern of -17 dB. The antenna achieved high front-to-Back (F/B) of 24.47 dB with gain variation of 0.55 dB over the operating bandwidth n258 (24.25 27.5) GHz .The proposed antenna has a wide operating bandwidth ranging from 23.33 GHz to 30 GHz.

Fig. 2 shown below presents the S_{11} for the parametric study steps over the n258 band for the proposed patch antenna for 5G applications at K – Band. The VSWR for proposed antenna shown below in Fig. 3.

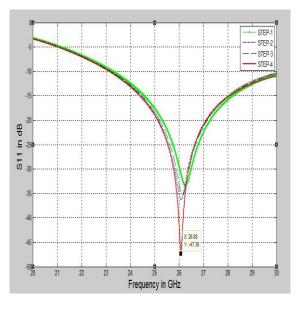
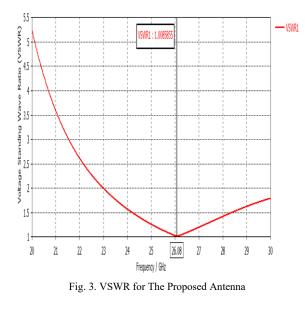


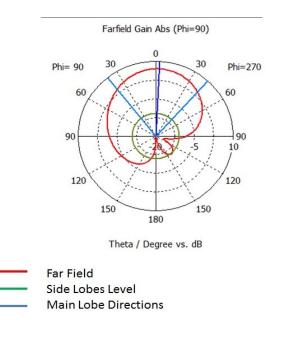
Fig. 2. Reflection Coefficient For Parametric Study of The Proposed Antenna Design



IV. DISCUSSION

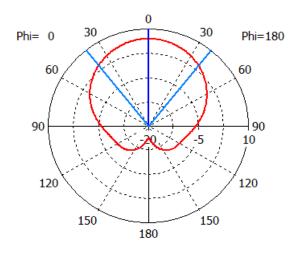
A microstrip patch antenna for 5G communications at K -Band was proposed in this article. The proposed antenna achieved gain of 8.10 dB which is higher than gain of conventional patch antenna. Moreover the proposed antenna has a far field radiation pattern with low SLL around -17 dB as shown in Fig. 4-(a) for Phi = 90 degrees and Fig. 4-(b) for Phi = 0 degrees . The proposed antenna achieved higher gain with lower physical area compared to antenna presented in [19] and a higher gain with smaller area compared to 2×1 MIMO antenna in [22]. The proposed 5G antenna achieved an operating bandwidth around 7GHz, which is wider than 0.380 GHz, which achieved by LI - slotted antenna presented in [14] with a comparable gain. Finally the proposed antenna achieved gain which higher than gain of 5G mm-wave with a smaller physical area [24]. Table-III presents a comparison between related works and proposed antenna achievements' in terms of gain and physical area of antenna. The proposed antenna achieved high F/B ration of 24.47 dB with a low gain variation over the n258 band, which found to be around 0.55 dB as shown in Fig. 5.

TABLE III Comparison Between proposed Antenna and Related Works					
Ref.	Gain (dB)	Physical Area (mm ²)			
[19]	2.60	11×8			
[22]	7.95	55×110			
[24]	6.10	8.6×9.2			
Proposed Antenna	8.10	8.2×8.5			



(a)

Farfield Gain Abs (Phi=0)



Theta / Degree vs. dB

(b)

Fig. 4. Far Field For The Proposed Antenna Design: (a) Phi = 90 and (b) Phi = 0 Degrees

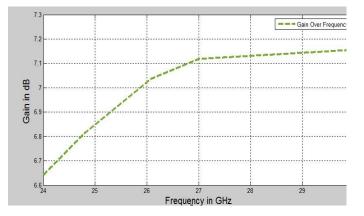


Fig. 5. Gain Over n258 Band for The Proposed Antenna.

V. CONCLUSION

A High gain and wide bandwidth patch antenna for fifth generation communication applications at K – Band was proposed. The antenna ;which was designed to operate at 26 GHz, is composed of a rectangular patch with three circles located above the rectangular shape with a specific space between this shape and each circle with two right angled triangles used to connect the three circles. The circular and triangular shapes had been added to achieve most optimum results in terms of gain, bandwidth, and reflection coefficient over the operating bandwidth n258 (24.25 - 27.50) GHz with low side lobes level far field.

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