

Pitfalls Associated with Using Dirac Distributions in the Area of Electronics and Telecommunications

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Abstract—This paper analyzes critically two graphical methods of representing sampled signals, which are exploited in the literature. Both are based on the description of the sampling operation with the use of a Dirac comb. However, they differ significantly from each other. It is explained here why this is the case and shown which of them is closer to reality. Reinterpretation of the first graphical representation discussed in view of the second one is also proposed. Finally, a remark about objects that are defined analytically but have no graphical representations is given.

Keywords—Graphical representations of sampled signals

I. INTRODUCTION

THE Dirac distribution, called also the Dirac impulse or the Dirac delta, was introduced into science in a rather intuitive way by P. A. M. Dirac [1], [2], a physicist, in order to solve problems of quantum mechanics that its creators were thinking about at that time, i.e. in the twenties of the last century. This mathematical tool, although lacking any rigorous foundation at the time of its appearance, proved however to be very effective, allowing to achieve results consistent with measurements carried out. In honor of its inventor, it now bears his name.

A mathematical justification for the correctness of the Dirac distribution was provided by L. Schwartz [3]. And it was probably R. Bracewell [4] who introduced this tool into the field of digital signal processing. He used it to describe mathematically the sampling operation of analog signals, performed in an ideal way. More precisely, he used for this purpose a comb of Dirac impulses operating on an analog signal and producing the so-called weighted Dirac comb with its weights being the amplitude samples of that signal. Further, we note that this approach to modelling sampling of analog signals and the resulting method of describing the sampled signals in a continuous (physical) time domain t has become well-established among people working in the field of digital signal processing. And nowadays, in fact, there is no other approach. To see that, it is enough to look into a few well-known textbooks and papers as, for example, [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

The essence of the model used there can be briefly characterized by stating that analog signal sampling is a specific type of modulation in which the carrier signal is a Dirac comb but the modulating one is the analog signal to be sampled. The

resulting signal of this modulation is a weighted Dirac comb, i.e., a sampled signal.

This interpretation of sampling operation as a specific type of modulation can be expressed analytically as follows:

$$x_{D,T}(t) = x(t) \cdot \delta_T(t) = \delta_T(t) \cdot x(t), \quad (1)$$

where $x(t)$ is an analog signal which is sampled (a modulating one). Furthermore, $x_{D,T}(t)$ in (1) stands for the sampled signal, i.e., the signal $x(t)$ after carrying out the sampling operation on it. Here, in our interpretation, it means performing the modulation of the Dirac comb, denoted by $\delta_T(t)$ in (1), with the use of the signal $x(t)$. The Dirac comb is defined as

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \quad (2)$$

where $\delta(t - kT)$, $k = \dots, -1, 0, 1, \dots$, stand for the time-shifted Dirac deltas. In (1) and (2), the letter T means the sampling period. Moreover, the index D in $x_{D,T}(t)$ stands for the name of Dirac.

Substituting (2) into (1) gives

$$x_{D,T}(t) = \sum_{k=-\infty}^{\infty} x(kT) \delta(t - kT), \quad (3)$$

where $x(kT)$'s are the samples of the signal $x(t)$ at the corresponding time instants kT 's.

In all the publications [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] cited above, the interpretation given by (1) with (2) and/or (3) has been used, either directly or indirectly.

There is however a certain problem with using (1) in signal processing theory. It follows from the fact that since the continuous time variable t occurs there in all three objects $x_{D,T}(t)$, $x(t)$, and $\delta_T(t)$ as an argument, it suggests that they can be treated as functions in any consideration that uses the sampling formula (1). And just in this way it is interpreted by the authors of the publications mentioned above and recommended to their readers for use. Further, it is precisely this reasoning that gave rise to the graphic illustration of a sampled signal, which is presented here in Fig. 1 and expressed analytically by (3). This interpretation of the sampled signal, as we know, is widely exploited in the literature.

The second issue that causes a lot of confusion in a mathematical theory of sampling and often leads to incorrect results is that the only symbolic notation

$$\int_{-\infty}^{\infty} \delta(t) \varphi(t) dt, \quad (4)$$

where $\varphi(t)$ denotes any so-called testing function correctly chosen according to the theory of distributions [3], [18], [19], [20], is commonly used in an extended interpretation (as a true integral) by engineers working in the field of digital signal processing. That is, not only as a symbolic notation, but also as a true mathematical expression.

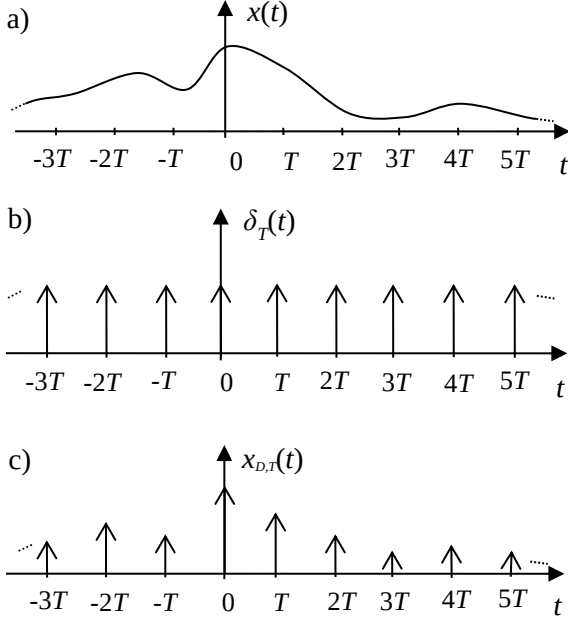


Fig. 1. Illustration of relation (1): a) modulating analog signal $x(t)$ – a signal to be sampled; b) Dirac comb $\delta_T(t)$ playing in (1) a role of a carrier signal – a signal which performs sampling of $x(t)$ uniformly on the continuous time axis Ot ; c) the weighted Dirac comb as a result of modulation of $\delta_T(t)$ by $x(t)$ – the sampled version of $x(t)$. Elements of this figure are taken from [21], [22], and [23].

In the next four sections, we explain in detail why the relation (1) (and/or (3)) and the notation (4) should not be used in a mathematical theory of sampling analog signals. Simply to avoid possible mistakes for sure. But if, despite the existence of such a danger, someone decides to use (1) and/or (3) should do this with great caution.

We continue our considerations in section II with taking a closer look at the notation (4) in the context of its use in the mathematical description of the sampling process. We highlight the reasons why it should not be used in a description of the sampled signal by a weighted Dirac comb. Further, in the material presented in the next sections III, IV, V and VI, we show what notation and definition is appropriate in this case, and how the sampled signal actually looks like. In the section VII, a useful note about objects that are well-defined analytically but have no graphical representations is given.

The paper ends with some conclusions and a few remarks.

II. MISLEADING NOTATION (4)

A question of the mathematical correctness of the object introduced to quantum mechanics by Dirac [1], and called by

him a delta function, is now fully explained. This happened thanks to works of L. Schwartz [3], [18], as we already know. He showed that it is a mathematical object which can be defined using the concept of a functional. And since such a functional can be used to define not only ordinary functions, but also objects that are not, all of them are called distributions [3], [18]. Or generalized functions, to indicate that they include both ordinary functions and objects such as the Dirac delta [3]. Or, to emphasize that the Dirac delta can be treated as a generalization of an ordinary function (to which L. Schwartz gave a precise mathematical meaning [3], [18]).

It must be admitted that the way of discussing and explaining the concept of a distribution, and in particular of the Dirac distribution, adopted in the textbooks devoted to these issues [18], [19], [20] does not help at all in understanding what it is. This is so because not only in the textbooks cited just above, but also in all others, the introduction to mathematics of this new object begins with defining it via a functional for ordinary functions. And this functional is a definite integral of the form: $\int_a^b f(t) \varphi(t) dt$ (with finite limits of integration a and b) or $\int_{-\infty}^{\infty} f(t) \varphi(t) dt$ (having infinite limits of integration), where $f(t)$ means a point-function but $\varphi(t)$ a properly chosen so-called testing function [18]. Both are functions of an independent variable t . Moreover, the integral mentioned can be considered in the sense of Riemann as well as in the sense of Lebesgue and it always exists for point functions [18].

Now, note that in order to move from the above definition of distribution specifying ordinary functions (point-functions) in a distributive manner to defining objects that are not such ones, we must abandon the definition of the functional using integral. Because then this integral simply does not exist. And this is what was done in all the textbooks mentioned above.

To see this necessity more clearly, let us take, for example, the integral given by (4), in which a point-function $f(t)$ is replaced by a Dirac delta $\delta(t)$. (In the rest of this paper, we restrict our considerations to only Dirac distributions so when we use the word distribution it means exclusively the Dirac distribution.) This integral does not exist either in the sense of Riemann or in the sense of Lebesgue without involving the Dirac measure in the latter case. Therefore, the defining functional must be defined differently in this case. And this is done with the help of the following notation [12], [19]:

$$D(\varphi) = \langle \delta(t), \varphi(t) \rangle = \varphi(0), \quad (5)$$

where $D(\varphi)$ means a functional, which, just by definition (5), gives the value $\varphi(0)$ for every properly defined testing function $\varphi(t)$. The symbol $\langle \delta(t), \varphi(t) \rangle$ represents another way [19] of writing the functional $D(\varphi)$, whose acting on functions $\varphi(t)$ returns one value $\varphi(0)$ in each case.

III. USE OF DEFINITION (5) OF DIRAC DISTRIBUTION IN DESCRIPTION OF OPERATION OF SAMPLING ANALOG SIGNALS

The mechanism of sampling an analog signal for description of the resulting one by means of Dirac deltas, used in the literature [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], is as follows. It is assumed that at a given time instant of sampling (we consider here a uniform sampling), the Dirac delta assigned to it “acts” impulsively on $x(t)$, returning the value of the analog signal amplitude at that instant. In other

words, the values of the signal amplitude at the successive sampling instants are returned by the “actions” of the corresponding Dirac deltas. And at the times outside of these instants, nothing is returned. Further, the Dirac delta assigned to the “action” to be performed at a given sampling instant, say kT , where $k = \dots, -1, 0, 1, \dots$, is the one whose “immediate action” assumed to occur at $t = 0$ (per basic definition) is now shifted on the timeline Ot to the point kT . So using (5), we can express this analytically as

$$D_{kT}(x) = \langle \delta(t - kT), x(t) \rangle = x(kT), \quad (6)$$

where $D_{kT}(\cdot)$ and $\delta(t - kT)$ stand for the functional defined by (5) and Dirac delta, respectively, with their “actions” shifted by kT on the continuous time axis Ot . $x(t)$ in (6) is an analog signal which is sampled and which satisfies all the conditions of a testing function $\varphi(t)$ in definition (5). Obviously, $x(kT)$ in (6) is a sample of $x(t)$ taken at the time instant kT .

Let us now denote the sampled signal by $y(t)$ as an object placed in the space of a physical time, i.e., of a continuous time t . Using the notation applied in (6), we can express it as

$$y(t) = \begin{cases} D_{kT}(x) = x(kT) & \text{for } t = kT, k \in \mathbb{Z}, \\ \text{undefined} & \text{for } t \text{ not equal to } kT, \end{cases} \quad (7)$$

where $\mathbb{Z}$ stands for the set of integers.

The sampled signal given by (7) is illustrated in Fig. 2.

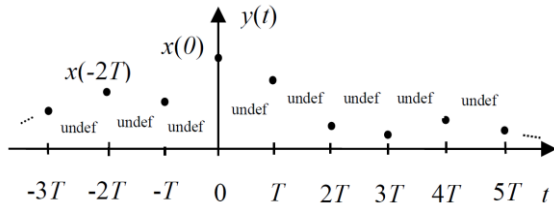


Fig. 2. Illustration of the sampled signal given by (7). Time intervals for which this signal is not defined are denoted by the abbreviation undef. The signal shown is a sampled version of $x(t)$ of Fig. 1a).

IV. COMPARISON OF DESCRIPTIONS (3) AND (7)

In the previous section, we arrived at a different description of the sampled signal (formula (7)) than the one given by formula (3), which is commonly used in the literature [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. It is evident that these two formulas differ diametrically from each other. This difference is also illustrated graphically here. The waveform of the sampled signal shown in Fig. 1c) with arrows going to infinity (according to (3)) looks completely different from the waveform in Fig. 2 (according to (7)). And they should, of course, look the same.

From the above, it follows that both the descriptions cannot be correct. And we have to decide which of them is correct. To this end, let us note that in devices that perform analog-to-digital conversion of signals (A/D converters), a finite number (finite sample value) always appears at their outputs at the sampling instants. At other times (between the successive sampling instants), there may be different numbers at their outputs (this is irrelevant). But usually these are the same numbers as at the beginning of each sampling period. So concluding, all this suggests that we must consider the sampled signal described by (7) to be its correct description. Not the description given by (3).

V. REMEDY FOR GRAPHICAL INTERPRETATION SHOWN IN FIG. 1C) OF DESCRIPTION GIVEN BY (3)

From a mathematical formal point of view, the analytical description of the sampled signal by (3) is, of course, correct if we assume that the sampling operation can be really described as presented in Section I: by modulating the Dirac comb with an analog signal, i.e., with the signal which is sampled – according to (1). However, recently doubts have arisen [21], [22], [23] as to the correctness of such a description, stemming from the lack of its coherence with the “physics and electronics” of a signal sampling process taking place in a real A/D converter. Here in Section IV, we have shown from a slightly different point of view that these doubts are justified.

In the next section, we show also that, regardless of the fact that the analytical expression (3) is incorrect, its graphical interpretation in Fig. 3c) is also defective. It is so because there exists no appropriate graphical interpretation of the Dirac distribution. Those illustrations of it that are shown in the literature are called naive representations in [20]. They can be treated only as substitutes of something that exists but cannot be drawn.

Further, it is also worth noting here that Fig. 1c) itself could be “saved” in the context of Fig. 2 and the associated description (7), when we would interpret its elements appropriately – so that they would correspond to those in Fig. 2. For this, the arrows in Fig. 1c) should have been then interpreted as vectors on the amplitude-time plane [24] with lengths equal to the values of the successive samples $x(kT)$, while the spaces on this plane between the straight lines passing vertically through the successive sampling points should have been interpreted as undefined (not with zero amplitude values, but just undefined). So it would seem that then $x_{D,T}(t)$ in Fig. 1c) is the same as $y(t)$ shown in Fig. 2.

Of course, this new interpretation of Fig. 1c) would not cause the interpretation of relations (1) and (3) to change. It would not change because these relations contain the Dirac deltas, not the Kronecker time functions [25], [26].

VI. MATHEMATICIANS’S POINT OF VIEW ON THE ISSUES RAISED IN THE PREVIOUS SECTIONS

The topic of a graphical representation of the Dirac distribution (Dirac delta), discussed in the previous sections of this paper, is certainly not an indifferent topic for both engineers and mathematicians. For engineers, it is not an indifferent topic because they do not know why in the model shown in Fig. 1, which is a prevalent one in the literature, they are supposed to assume that the samples in a sampled signal occur in form of the weighted Dirac distributions instead of what they actually see at the output of an A/D converter. And further, they receive graphical interpretations of these distributions in form of vertical arrows starting on the timeline Ot and running up to infinity, as we can see in Fig. 1c). But what do they see in fact instead? They see numbers that do not disappear immediately but retain their “validity” throughout the entire sampling period T . Or, as in radio-photonics systems [27], they see very narrow weighted pulses (which can be modelled well by Gaussian bell functions) appearing during sampling periods.

Of course, the above two descriptions are distinct, which is frustrating for engineers; they would expect a description of the sampled signal and a graphical interpretation of it that would be easily understandable.

The topic of a graphical interpretation of the Dirac distribution is also not indifferent to mathematicians because they have set unconsciously many traps for engineers regarding this topic and, as it turns out, they are not very able to get out of them. And in sciences is such a belief that if an engineer cannot describe something well in mathematical terms, she/he goes to a mathematician who can always do it well. But it does not work here, so it seems worth to analyze why this happens; we do this shortly below.

Perhaps a general remark should be as follows: introducing the Dirac distribution into the description of sampling operation is somewhat of a misunderstanding. To describe the result of sampling a continuous-time signal, the description as shown in Fig. 2 is fully sufficient. It contains only the sample values at the sampling instants; there are no Dirac distributions.

Some argue that the introduction of the Dirac deltas into the mathematical sampling theory was necessary in order to be able to compute the Fourier transforms of sampled signals. But this is a false claim, since it is well known that Poisson's identity [12] suffices for this purpose. It relates the Fourier transform of the sampled signal, defined as the so-called Discrete Time Fourier Transform (DTFT) [12], [13], [14], with a sum of the frequency-shifted Fourier transforms of the analog signal before its sampling.

VII. NOTE ON OBJECTS THAT ARE DEFINED ANALYTICALLY BUT HAVE NO GRAPHICAL REPRESENTATIONS

It has been mentioned in section V that the interpretation of Dirac deltas by means of arrows running to infinity, as shown in Fig. 1c), is defective. But this is not an isolated case in the fields of electronics and telecommunications (as well as in mathematics), where the analytical description of an object is well-defined but cannot be represented graphically. We would like to make this fact clear to the readers, albeit briefly in this section, to remove the impression that the lack of a proper graphical representation of the Dirac delta is something extraordinary. In addition to the Dirac delta, we discuss also here two further examples: constellations of the M-FSK digital modulations [28], where $M = 4, 8, 16$ etc., and the Dirichlet function [29]. They have either no graphical representations at all in one-, two- or three-dimensional geometrical spaces or only substitute constructs are used for them.

Why is it difficult to deduce a graphical representation (agraphical illustration) of the Dirac delta $\delta(t)$ from such a clear and transparent analytical relation (5)? This is so because it is an ambiguous object in the sense that there are many possible ideas as to what it might look like, when leading to the result $\varphi(0)$. The most common approach in the digital signal processing literature [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] is to visualize this object as an increasingly narrower time pulse with an area equal to unity at each case, despite its decreasing width, which in the limit becomes one of those arrows extending indefinitely, as shown in Fig. 1b). Another way to visualize it could be by invoking the graph of the Kronecker function [25], [26] (consisting of only one 1 at a specific time point, here for $t = 0$), multiplied by the function $\varphi(t)$. The result of this operation would be equal to $\varphi(0)$ [25], [26]. Another way of visualization would potentially offer an operator that "pulls out" one value from the function $\varphi(t)$ at a specific time instant. But how to show it graphically?

It follows from the above that the graphical representation of the Dirac delta used in the literature [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], e.g. to visualize a sampled

signal (as shown in Fig. 1c), is merely a substitute for something that is not determinable (the graphical representation of $\delta(t)$). Of course, unlike a clearly defined analytical definition of $\delta(t)$ given by (5).

In the theory of digital modulations, the digital modulations ASK (Amplitude-Shift Keying), PSK (Phase-Shift Keying), FSK (Frequency-Shift Keying) and QAM (Quadrature Amplitude Modulation) are described and explained with the use of geometric concepts, i.e. in the so-called signal spaces [28], [30], [31]. The modulated signals at the output of a given type of modulators listed above are linear combinations of the orthonormal signals (called basic signals), which define mutually orthogonal directions in the aforementioned signal spaces. And each of these signals can be considered as a vector in the corresponding space for a given modulation type. Further, the vectors are uniquely represented by the coefficients in their associated linear combinations of the basic orthonormal signals, and are represented graphically by points on graphs called constellation diagrams.

For the different types of digital modulations mentioned above, the dimensions of the associated signal spaces may (but do not have to) differ. Thus, in the case of ASK modulation, it will always be a one-dimensional space (associated with a single carrier $\theta(t) = (\sqrt{2/T_b})\cos(2\pi f_c t)$, where T_b means the bit duration and f_c the carrier frequency; $\theta(t)$ is here a basic function), regardless of the number of points on its constellation diagram [28]. In the case of BPSK modulation, which means a two-point modulation, i.e. a Binary Phase Shift-Keying modulation, a one-dimensional signal space is also sufficient for its description. However, if there are more than two points, they will be distributed in a two-dimensional space (associated with two carriers: the in-phase $\theta_1(t) = (\sqrt{2/T_b})\cos(2\pi f_c t)$ carrier and the quadrature $\theta_2(t) = (\sqrt{2/T_b})\sin(2\pi f_c t)$ one). For illustration, Fig. 3 shows three PSK constellation diagrams: for BPSK, for 4-PSK (four-point one), and for 8-PSK (eight-point one).

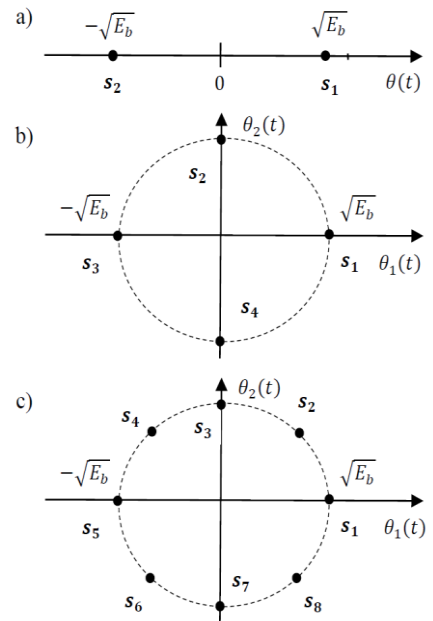


Fig. 3. Constellation diagrams for some PSK modulations: a) for BPSK, b) for 4-PSK, and c) for 8-PSK. The constellation points in figure are marked by black points and are denoted as s_1 and s_2 for BPSK; s_1, s_2, s_3 , and s_4 for 4-PSK; and $s_1, s_2, s_3, s_4, s_5, s_6, s_7$, and s_8 for 8-PSK. E_b means here the transmitted signal energy per bit.

The situation is similar in QAM modulations. All of them (starting with that having the smallest number of points, i.e., four) are two-dimensional, which means that their constellation diagrams always consist of points lying on a plane.

One- or two-dimensional spaces are not however sufficient for the descriptions of FSK modulations. Only in the case of a Binary Frequency-Shift Keying (BFSK) modulation, where the number of points in its signal space is equal to two, one can visualize what its constellation diagram looks like and draw it. This is shown in Fig. 4.

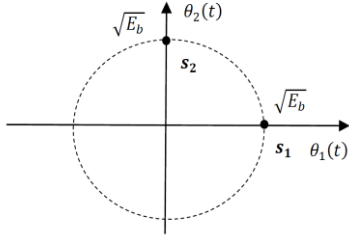


Fig. 4. Constellation diagram for the BFSK modulation.

In the BFSK scheme, basic signals are cosinusoids: $\theta_1(t) = (\sqrt{2/T_b})\cos(2\pi f_{c1}t)$ and $\theta_2(t) = (\sqrt{2/T_b})\cos(2\pi f_{c2}t)$, with their frequencies f_{c1} and f_{c2} chosen in such a way that these signals are orthogonal to each other. Therefore, they determine the orthogonal directions in a two-dimensional space visualized in Fig. 4. So the points $\mathbf{s}_1$ and $\mathbf{s}_2$ shown in this figure have an equivalent vector form with the coordinates $\sqrt{E_b}, 0$ and $0, \sqrt{E_b}$, respectively. That is, these vectors can be expressed as $\mathbf{s}_1 = [\sqrt{E_b} \ 0]$ and $\mathbf{s}_2 = [0 \ \sqrt{E_b}]$ (as row vectors).

Let us consider now, for example, 4-point and 8-point FSK modulations, called 4-FSK and 8-FSK in short. In the case of the former, we have the following four basic functions: $\theta_1(t) = (\sqrt{2/T_s})\cos(2\pi f_{c1}t)$, $\theta_2(t) = (\sqrt{2/T_s})\cos(2\pi f_{c2}t)$, $\theta_3(t) = (\sqrt{2/T_s})\cos(2\pi f_{c3}t)$, and $\theta_4(t) = (\sqrt{2/T_s})\cos(2\pi f_{c4}t)$, with their frequencies f_{c1}, f_{c2}, f_{c3} , and f_{c4} chosen in such a way that these signals are orthogonal to each other. Therefore, they determine the orthogonal directions in a four-dimensional space. Whereas the corresponding points of the 4-FSK constellation have the form [28], [30], [31]: $\mathbf{s}_1 = [\sqrt{E_s} \ 0 \ 0 \ 0]$, $\mathbf{s}_2 = [0 \ \sqrt{E_s} \ 0 \ 0]$, $\mathbf{s}_3 = [0 \ 0 \ \sqrt{E_s} \ 0]$, and $\mathbf{s}_4 = [0 \ 0 \ 0 \ \sqrt{E_s}]$, where E_s stands for the transmitted signal energy per symbol. And T_s in the basic functions given above means the symbol duration.

Note that it follows from the above clearly that, analytically, the constellation of points in the 4-FSK modulation is precisely and uniquely determined by the four vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$, and $\mathbf{s}_4$ defining the orthogonal directions indicated by the basic functions $\theta_1(t), \theta_2(t), \theta_3(t)$ and $\theta_4(t)$, in the four-dimensional space. The problem here, however, is that it is difficult for us to imagine how this distribution looks like in such a space, since our imagination can only grasp spaces with dimensions fewer than four. Of course, as done above, there is no problem with writing down all of this analytically. But we state once again that due to our human limitations we cannot visualize it, let alone draw it, even using the tools of the projective geometry.

To further illustrate the problem with the constellation diagrams for M-point FSK modulations, when $M > 3$, let us consider here yet the 8-FSK modulation. In this case, we have

the following eight basic functions:

$$\begin{aligned} \theta_1(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c1}t), & \theta_2(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c2}t), \\ \theta_3(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c3}t), & \theta_4(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c4}t), \\ \theta_5(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c5}t), & \theta_6(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c6}t), \\ \theta_7(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c7}t), & \text{and } \theta_8(t) &= (\sqrt{2/T_s})\cos(2\pi f_{c8}t) \end{aligned}$$

with their frequencies $f_{c1}, f_{c2}, f_{c3}, f_{c4}, f_{c5}, f_{c6}, f_{c7}$, and f_{c8} chosen in such a way that these signals are orthogonal to each other. Therefore, they determine the orthogonal directions in an eight-dimensional space. Whereas the corresponding vectors (points) of the 8-FSK constellation have the form [28], [30], [31]:

$$\begin{aligned} \mathbf{s}_1 &= [\sqrt{E_s} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], & \mathbf{s}_2 &= [0 \ \sqrt{E_s} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0], \\ \mathbf{s}_3 &= [0 \ 0 \ \sqrt{E_s} \ 0 \ 0 \ 0 \ 0 \ 0], & \mathbf{s}_4 &= [0 \ 0 \ 0 \ \sqrt{E_s} \ 0 \ 0 \ 0 \ 0], \\ \mathbf{s}_5 &= [0 \ 0 \ 0 \ 0 \ \sqrt{E_s} \ 0 \ 0 \ 0], & \mathbf{s}_6 &= [0 \ 0 \ 0 \ 0 \ 0 \ \sqrt{E_s} \ 0 \ 0], \\ \mathbf{s}_7 &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \sqrt{E_s} \ 0], & \mathbf{s}_8 &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \sqrt{E_s}]. \end{aligned}$$

It follows from the above analytical description that the constellation of points in the 8-FSK modulation is precisely and uniquely determined by the eight vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5, \mathbf{s}_6, \mathbf{s}_7$, and $\mathbf{s}_8$ defining the orthogonal directions indicated by the basic functions $\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t), \theta_5(t), \theta_6(t), \theta_7(t)$, and $\theta_8(t)$, in the eight-dimensional space. So we see clearly that the problem of visualizing and drawing a constellation diagram in this case is far more challenging. Evidently, this task is impossible to be performed.

Our third example, we consider in this section is the so-called Dirichlet function [32], [33]. It is defined in the following way:

$$DIR(t) = \begin{cases} 1 & \text{for } t \in \mathbb{Q} \\ 0 & \text{for } t \notin \mathbb{Q} \end{cases}, \quad (8)$$

where $\mathbb{Q}$ means the set of rational numbers. This function is an example of a function that is discontinuous everywhere. It can be also expressed [32] as

$$DIR(t) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \cos^{2n}(m! \pi t). \quad (9)$$

Furthermore, $DIR(t)$ is a characteristic function of the set of rational numbers (which are everywhere intertwined with the irrational ones). All of this means that this function cannot be drawn. So in other words, it has no graphical representation.

Sometimes, one comes across a graph of the function $DIR(t)$ as shown in Fig. 5. Note, however, that this graph cannot be considered a true graphical representation of the function $DIR(t)$ since both the lower and upper blue lines (branches of this graph) appear to be continuous straight lines, whereas in reality each of them should show an infinite number of "gaps" (of different set cardinalities, infinite in both cases). So this graph can only be treated as a substitute of a graphical representation that cannot be drawn.

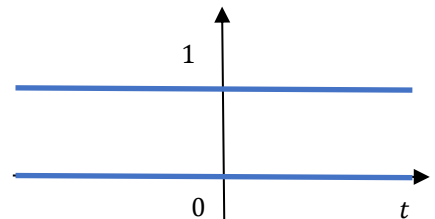


Fig. 5. Graph (substitute) of the Dirichlet function.

VIII. CONCLUSIONS AND REMARKS

Once again, by analyzing the graphical representation of the sampled signal, which is commonly used in the literature, it has been shown here that it is not appropriate. Even more, it can be said that it is wrong. A correct graphical form of the discrete signal is presented in Fig. 2, and given analytically by (7).

The only thing in Fig. 2 that is in any way related to the Dirac deltas are the values of samples of the analog signal, which, if one wishes, can be calculated with the use of the incorrect formula (3). But really, there is no need to do so, since they are simply numbers, which do not need any conversion to some other objects. As, for example, let us take a signal sample $x(2T)$. It is a number and it does not need to be converted into the distribution $x(2T)\delta(t - 2T)$. And if someone does it, it is already a manipulation of a physical quantity. That is, converting the number $x(2T)$ into the distribution $x(2T)\delta(t - 2T)$. Such operations can lead to incorrect results in analyses using distributions $x(kT)\delta(t - kT)$ instead of samples $x(kT)$.

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