

Robust TDoA localization via Quadratic Weighted Least Squares in degenerate geometries

Igor Kravets, Ostap Shuparsky, and Andriy Luchechko

Abstract—This letter presents a zone-optimized Quadratic Weighted Least Squares (QWLS) algorithm for time-difference-of-arrival (TDoA)-based localization using wireless sensors. QWLS extends previously developed Parametric Three-Stage Weighted Least Squares (P3WLS) method by introducing a generalized formulation that remains robust under a broad range of eigenvalue conditions. Unlike P3WLS, which fails when the smallest eigenvector of the system matrix becomes nearly tangent to the TDoA constraint surface, QWLS introduces a second-order correction in the eigenbasis to enforce tangency and maintain physical consistency. We focus on the specific “worst-case” zones where P3WLS becomes unstable due to near-degenerate geometry and demonstrate that QWLS significantly improves localization accuracy in these regions. Simulation results across a mixed indoor-outdoor sensor deployment show that QWLS consistently outperforms both standard Weighted Least Squares and the original P3WLS method in structured failure zones

Keywords—TDoA-based localization, Wireless sensors, Quadratic weighted least squares, Constraint-based estimation

I. INTRODUCTION

ULTRA wideband (UWB) sensor networks enable high-precision short-range positioning in industrial settings. A typical deployment uses fixed anchors and mobile tags. Time-Difference-of-Arrival schemes scale better than ToA because sensors avoid per-anchor handshakes, which reduces channel load and extends battery life. Modern UWB devices provide sub-nanosecond timing that supports precise TDoA even in cluttered environments [1]. Because tags and anchors are not synchronized, localization requires solving a nonlinear system that couples position and clock bias through a quadratic range-consistency constraint [2], [3]. In practice, Weighted Least Squares (WLS) methods are standard for handling noisy, redundant measurements [4], [5], [6], [7], [8]. Conventional two-stage WLS (2WLS) solves an unconstrained linear approximation and then refines with fixed weights, but accuracy degrades in near-degenerate TDoA geometries where the Gram matrix approaches singularity [9], [10]. Remedies include Tikhonov ridge regularization [11] and eigenvector-guided refinements such as constrained projection or trust-region updates [4]. Among targeted approaches, the Parametric Three-Stage WLS (P3WLS) [10] searches along the smallest-eigenvector direction and intersects that line with the quadratic

constraint which works well under moderate ill-conditioning. But a distinct worst-case zone arises when the smallest eigenvector is nearly tangent to the constraint surface. Tiny perturbations then deviate the line-conic intersection, producing large errors. This reflects the general instability of methods that ignore second-order curvature near singular points [12]. We therefore introduce a Quadratic WLS (QWLS) estimator that models equal-error increases as an ellipsoidal surface around the unconstrained solution and enforces tangency between this ellipsoid and the quadratic constraint. The resulting estimate remains well-posed as the smallest eigenvalue tends to zero. While broadly applicable, we target the P3WLS failure zones and show in simulation that QWLS consistently outperforms 2WLS and P3WLS there. Following a review of the P3WLS method, the main contributions of this work are as follows: (i) geometric analysis of the P3WLS failure zone; (ii) a QWLS estimator that uses the full eigen-structure while satisfying the nonlinear range constraint; and (iii) a simulation study demonstrating robustness gains in challenging geometries.

II. P3WLS ALGORITHM

P3WLS [10] algorithm augments 2WLS with a constrained one-dimensional search to handle near-singular geometries.

A. First Step

So basic equation of P3WLS first step is:

$$Ap = b$$

where

$$A = 2 \begin{bmatrix} \bar{x}_{i1} & \bar{y}_{i1} & c\tau_{i1} \end{bmatrix}$$

$$b = \begin{bmatrix} x_{21}^2 + y_{21}^2 - (c\tau_{21})^2 \\ \vdots \\ x_{i1}^2 + y_{i1}^2 - (c\tau_{i1})^2 \end{bmatrix} \quad p = \begin{bmatrix} x & y & d \end{bmatrix}^T$$

Here x_{i1}, y_{i1} denote the relative positions of the i^{th} receiver sensor with respect to the first receiver, τ_{i1} represents the TDoA between the i^{th} receiver sensor and the first. The vector p corresponds to the unknown coordinates of the transmitter, and d is a dummy variable introduced to linearize the range constraint. To obtain an explicit solution, a standard least-squares estimation is applied:

$$\hat{p}_1 = (A^H A)^{-1} A^H b = G^{-1} A^H b \quad (1)$$

where $G = A^H A$ is the Gram matrix.

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B. Second Step

Under degenerate geometries, the Gram matrix G can become nearly singular, so that can lead to large displacement errors in the estimated position. To address this, the second step of the P3WLS algorithm fits the quadratic constraint by parametrically constructing a span of closest solutions:

$$E(p) = \varepsilon^T \varepsilon = (Ap - b)^T (Ap - b)$$

or

$$E(p) = E(\hat{p}_1) + (p - \hat{p}_1)^T G(p - \hat{p}_1) \quad (2)$$

where $E(p)$ is least-squares objective function to be minimized, ε is the least-squares residual in $\mathbb{R}^3$, and $E(\hat{p}_1)$ is the minimized cost calculated from Equation (1).

In case where G is nearly singular, we construct a parametric family of solutions $\hat{p}_2$ defined with respect to a scalar parameter t :

$$\hat{p}_2(t) = \hat{p}_1 + v_k \cdot t \quad (3)$$

where v_k is the eigenvector of the Gram matrix G corresponding to the smallest eigenvalue $\lambda_k \rightarrow 0$ in degenerate geometry.

C. Third Step

After calculating the span of possible solutions, the next step is to apply the conic constraint associated with the dummy variable d :

$$f(p) = p^T B p = 0 \quad \text{or} \quad x^2 + y^2 = d^2 \quad (4)$$

where, $B = \text{diag}(1, 1, -1)$

Equation (4) defines a conical surface in $\mathbb{R}^3$. By substituting the parametric solution $\hat{p}_2$ from Equation (3) into the constraint, we obtain:

$$(\hat{p}_1 + v_k \cdot t)^T B (\hat{p}_1 + v_k \cdot t) = 0$$

If the Gram matrix G has only one near-zero eigenvalue, the scalar t becomes the sole unknown, resulting in 2 unique solutions.

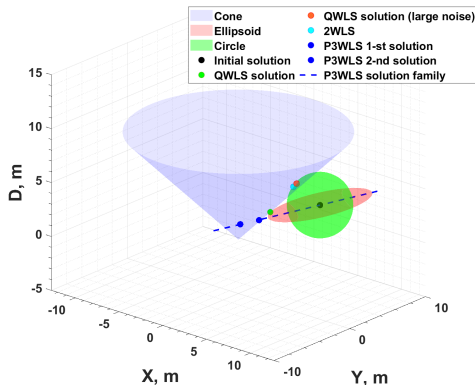


Fig. 1: Solutions topology. The P3WLS family of solutions becoming inaccurate due to elliptic surface curvature. In such cases, the Quadratic Weighted Least Squares (QWLS) method provides a more robust alternative

P3WLS often outperforms 2WLS, even in well-conditioned cases, but it assumes a near-singular geometry matrix [10]. In symmetric layouts (e.g., a square room with anchors at the corners), a degradation zone can appear near a corner: the smallest eigenvector becomes nearly tangent to the conic constraint, making the line-conic intersection unstable (Fig. 1). Such geometries are common in practical UWB deployments, and small timing perturbations from multipath or hardware noise can push P3WLS beyond its effective range. These limitations motivate QWLS, which maintains accuracy across a broader set of geometries.

III. QWLS ALGORITHM

This section presents the Quadratic Weighted Least Squares (QWLS) as a nonlinear constrained quadratic programming (NCQP) problem solvable via standard numerical optimization. While the original TDoA equations form a highly nonlinear, nonconvex system requiring costly, initialization-sensitive iterations [6], [8], the NCQP-applied after P3WLS's first step works in the geometry matrix's eigenspace, reducing dimensionality while preserving physical consistency, minimizing least-squares error, and improving efficiency over solving the full TDoA system.

The quadratic term in Equation (3) defines a second-order surface in p -space. Since the Gram matrix $G \succcurlyeq 0$ is a positive semidefinite, the shape of this surface depends on the number of zero eigenvalues in the matrix spectrum. Specifically:

$$\mathfrak{S}(k) = \begin{cases} \text{ellipsoid,} & k = 0, \\ \text{cylindrical (one zero eigenvalue),} & k = 1, \\ \text{disk / degenerate ellipsoid,} & k = 2. \end{cases}$$

Here, k is the number of zero eigenvalues, and $\mathfrak{S}(k)$ denotes the corresponding surface form. In our previous work [10], we analyzed the case $k = 1$, where the resulting cylindrical surface allowed P3WLS to significantly outperform 2WLS, even under this coarse eigenspectrum approximation.

Under high noise conditions, we expect the ellipsoidal surface to collapse into a near-spherical form (Fig. 1). This occurs because perturbations in the initial estimate and eigenvalues tend to symmetrize the shape of the error surface, making it approximately invariant to rotation in p -space.

To improve robustness in such scenarios, we apply Ridge Regression (Tikhonov regularization) [11] to the eigenspectrum of the Gram matrix. This accounts for noise by inflating the eigenvalues and incorporating uncertainty directly into the LS surface. The Gram matrix is first diagonalized as:

$$G = V \Lambda V^T$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, contains eigenvalues, and $V = [v_1, v_2, v_3]$ is the matrix of corresponding eigenvectors. To regularize the matrix, we apply a small positive shift to the eigenspectrum:

$$\Lambda' = \Lambda + \eta I$$

where Λ' are the regularized eigenvalues, η is a tuning parameter based on the expected noise level, and $I \in \mathbb{R}^3$ is the identity matrix. This regularization scheme improves stability and allows tuning based on environmental calibration. The application of Gram matrix regularization will be explored further in future work, with a focus on environment-aware tuning, reducing computational complexity, and integrating hybrid methods to improve proximity tracking accuracy.

To simplify the solution process, we reparametrize the vector p in polar coordinates constrained to the cone:

$$p = \begin{bmatrix} r \cos(\phi) \\ r \sin(\phi) \\ r \end{bmatrix}$$

This ensures that the solution vector p lies on the conic constraint, Equation (4). So, the final optimization problem becomes:

$$\min_{r, \phi \in \mathbb{R}} (p - \hat{p}_1)^T G (p - \hat{p}_1)$$

$$\text{subject to: } r > 0, 0 \leq \phi < 2\pi$$

The NCQP can be solved with standard nonlinear optimizers - SQP, SNOPT, KNITRO, and trust-region methods (Tron, Trust-C) [12], [13], [14], [15]. We use gradient descent method as a proof-of-concept to validate QWLS feasibility and robustness under challenging geometries; although too heavy for real-time, their performance confirms the formulation's soundness and motivates closed-form or low-complexity QWLS-specific iterations for embedded platforms.

IV. SIMULATION

In practical applications, TDoA-based localization systems are subject to various perturbations, with Gaussian noise being the most prevalent [1], [3]. Such noise typically arises from thermal electronic fluctuations or external radio interference.

The noiseless TDoA is computed as the difference between the arrival times at the i^{th} and a selected reference j^{th} sensor:

$$\tau_{ij} = t_i - t_j$$

For this study, we assume all sensors experience equal Gaussian noise variance, leading to a simplified model:

$$\Delta\tau_{ij} \sim \mathcal{N}(0, \sigma_{TDoA}^2)$$

In our simulations, we consider standard deviations of 10 ps, 100 ps, and 1 ns, which correspond approximately to 3 mm, 3 cm, and 30 cm of range error, respectively, assuming propagation at the speed of light. The experimental environment models a square room measuring 10x10 meters, with receiver sensors placed in the corners.

A. Simulation Results and Analysis

This section presents the results of performance simulations for the evaluated localization algorithms.

Fig. 3 evaluates the sensitivity of the proposed algorithms to varying levels of Gaussian time jitter, modeled as normally

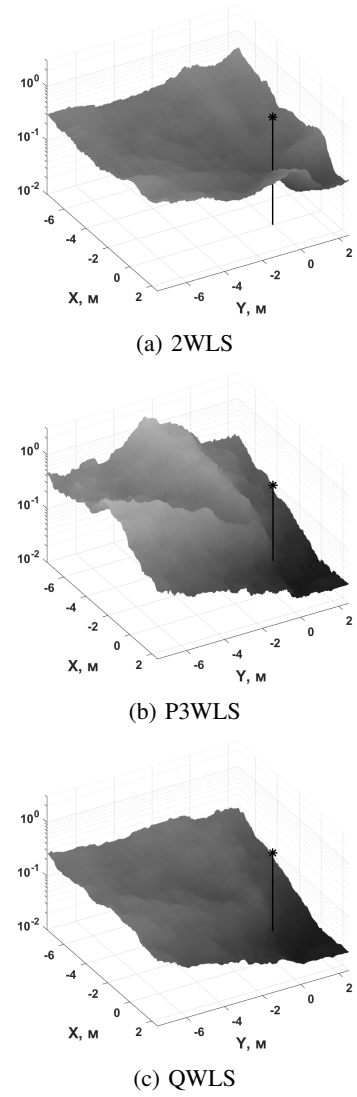


Fig. 2: Root-mean-square (RMS) of the absolute displacement between the ground truth transmitter position and the estimated position for each algorithm **for P3WLS worst case area**, plotted across spatial locations. Simulations were performed with a Gaussian noise standard deviation of $\sigma = 100ps(\approx 3cm)$. At each transmitter location, 100 independent measurements were averaged over 20,000 spatial positions.

distributed perturbations that reflect realistic UWB sensor performance.

Across all noise regimes, QWLS consistently demonstrates superior robustness. This is particularly evident in the high-precision domain, where P3WLS becomes increasingly unstable due to near-degenerate geometries. Notably, high white noise levels prevent the Gram matrix G from becoming singular, and the initially elongated ellipsoidal error surface becomes more sphere-like due to the isotropic nature of Gaussian noise in space.

Fig. 3 demonstrates that the QWLS algorithm consistently outperforms the conventional 2WLS approach across the full range of Gaussian noise levels. It also significantly improves upon the single-eigenvector-based P3WLS method by incor-

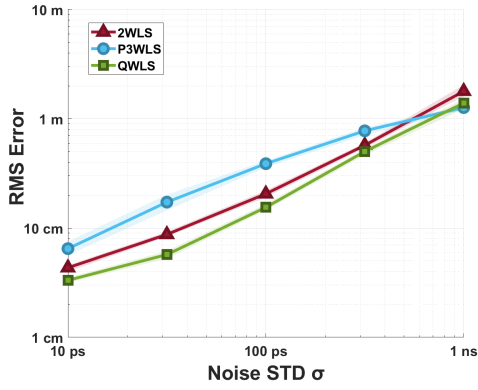


Fig. 3: Root-mean-square (RMS) of the absolute displacement between the ground truth transmitter position and the estimated position **for P3WLS worst case area**, plotted as a function of Gaussian noise standard deviation for each algorithm. The shaded region indicates the 95% confidence interval based on repeated measurements

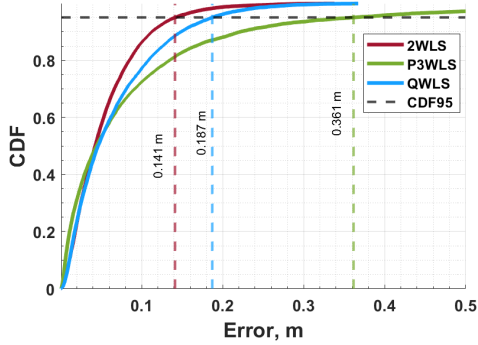


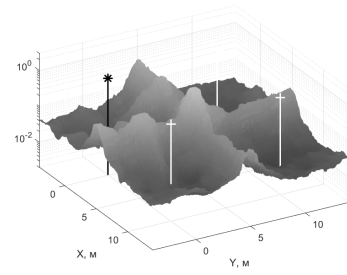
Fig. 4: RMSE CDF plot **for P3WLS worst case area**

porating the full curvature of the least-squares (LS) error equipotential surface. Under low and moderate noise conditions, QWLS achieves noticeably lower root-mean-square error (RMSE) and tighter confidence intervals - showing up to a 200% reduction in RMSE compared to P3WLS. However, under high-noise conditions, the reliability of the QWLS eigenvalue-based curvature model decreases. The error surface becomes more isotropic, and the method's performance slightly degrades, showing approximately 9% higher RMSE than P3WLS. This limitation can potentially be mitigated by introducing the tuning parameter η , as discussed in Section III, to regularize the eigenspectrum based on expected noise levels.

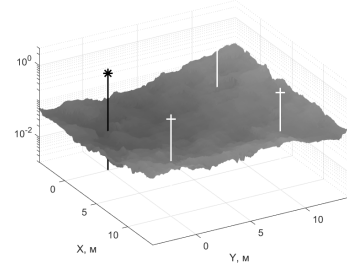
TABLE I: Worst case zone RMSE statistics, cm

Algorithm	MEAN	RMS	STD	CDF50	CDF95
2WLS	5.4	7.1	4.6	4.2	14.1
P3WLS	9.3	17.3	14.6	4.4	36.1
QWLS	6.6	8.9	6.0	4.6	18.7

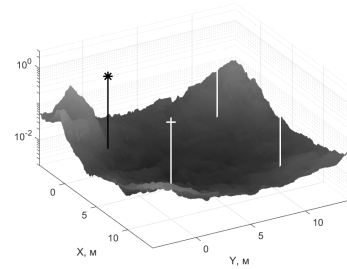
Fig. 2 illustrates the comparison of the methods in the critical failure zone of P3WLS. In this region, where the smallest eigenvector becomes nearly tangent to the conic



(a) 2WLS



(b) P3WLS



(c) QWLS

Fig. 5: Root-mean-square (RMS) of the absolute displacement between the ground truth transmitter position and the estimated position for each algorithm **for whole anchor deployment area**, plotted across spatial locations. Simulations were performed with a Gaussian noise standard deviation of $\sigma = 100ps (\approx 3cm)$. At each transmitter location, 100 independent measurements were averaged over 20,000 spatial positions.

constraint, QWLS exhibits visibly improved robustness. The overall performance could be further enhanced through more accurate NCQP solvers, larger sample sizes, or hybrid techniques incorporating geometry-aware criteria such as the α and λ metrics proposed in our earlier work [10].

TABLE II: Whole anchor placement zone RMSE statistics, cm

Algorithm	MEAN	RMS	STD	CDF50	CDF95
2WLS	35.6	1948.2	1948.0	4.0	46.0
P3WLS	2.6	5.4	4.7	1.2	9.2
QWLS	8.8	11.9	7.9	6.4	24.7

However, Fig. 4 and Table I show that QWLS does not outperform 2WLS algorithm at all metrics, but its efficiency is close to 2WLS.

Fig. 5, 6 and Table II display the general efficiency of the algorithms across the whole range of anchor configuration.

2WLS is expectedly the worst, because of the ill-conditioned G matrix. P3WLS dominates at the inside zone, outperforming both QWLS and 2WLS. So QWLS could be used as the non-hybrid alternative, that doesn't require any additional criteria-based method switching.

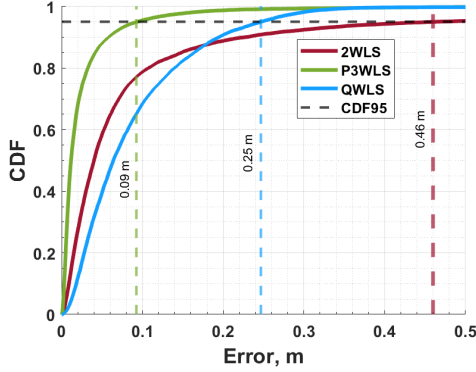


Fig. 6: RMSE CDF plot for whole anchor deployment area

V. EXPERIMENT

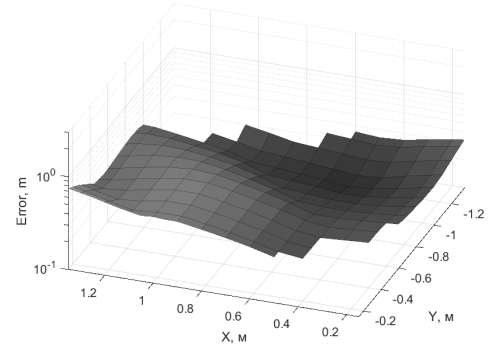
Experiment took place in underground parking lot (Fig. 7), complex environment with many reflection sources such as metal columns, pipes, concrete floor and ceiling. Worst case zone for P3WLS algorithm was studied, closest few meters behind the reference anchor in 10x10 m anchor configuration. 'Ground Truth' data by definition is measured by laser distance meter, so accuracy of the GT definition is in millimeter range. SynchronicIT SFM10-MOD UWB modules were used as devices in the experiment. Fig. 8, 9 and Table III display performance of the algorithms.



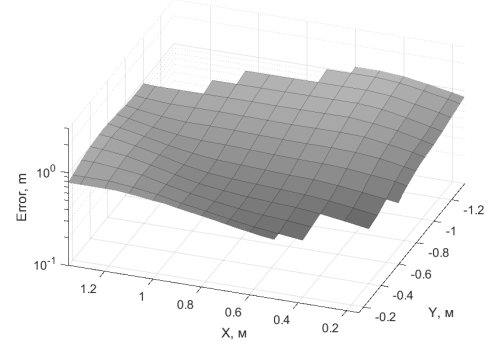
Fig. 7: Experiment environment. Underground parking lot

VI. CONCLUSION

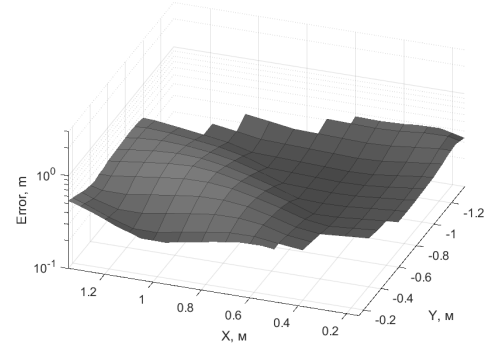
This letter presented the Quadratic Weighted Least Squares (QWLS) algorithm as a robust generalization of the Parametric Three-Stage Weighted Least Squares (P3WLS) method for TDoA-based localization using UWB sensors. While P3WLS improves accuracy in ill-conditioned geometries, it becomes



(a) 2WLS



(b) P3WLS



(c) QWLS

Fig. 8: Root-mean-square (RMS) of the absolute displacement between the ground truth transmitter position and the estimated position for each algorithm obtained from experiment, plotted across spatial locations

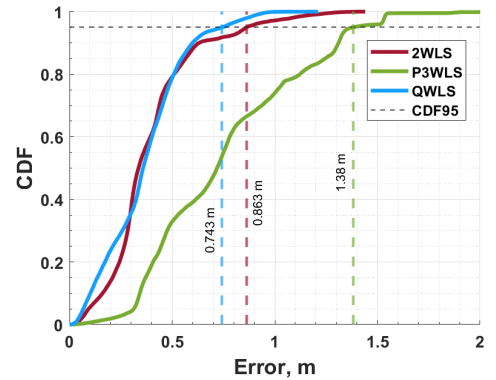


Fig. 9: RMSE CDF plot obtained from experiment

TABLE III: Underground parking lot experiment RMSE statistics, cm

Algorithm	MEAN	RMS	STD	CDF50	CDF95
2WLS	39.0	44.9	22.2	33.6	86.3
P3WLS	75.9	84.3	36.9	71.8	138.1
QWLS	36.3	41.3	19.7	35.8	74.3

unstable in specific worst-case zones where the smallest eigenvector of the geometry matrix becomes nearly tangent to the conic constraint surface. QWLS overcomes this limitation by leveraging the full eigenspectrum and enforcing a tangency condition between the least-squares error ellipsoid and the nonlinear range-consistency constraint.

Rather than solving the original nonlinear TDoA system directly, which is known to be nonconvex, computationally intensive, and sensitive to initialization, QWLS operates within a reduced-dimensional eigenbasis derived from the initial P3WLS step. This formulation transforms the problem into a nonlinear constrained quadratic programming (NCQP) task that retains physical consistency, minimizes the least-squares error, and is significantly more efficient to solve using standard optimization techniques.

Simulation results under varying levels of Gaussian noise confirm that QWLS mostly outperforms P3WLS, but slightly falls behind two-stage weighted least squares (2WLS) in structured failure regions. The proposed methods remain robust even in near-degenerate configurations, offering a reliable framework for accurate localization in challenging environments. QWLS is proposed to be switchless alternative to the 2WLS and P3WLS algorithm, displaying moderate efficiency without any need to rely on additional criteria.

Experiment results portrayed better QWLS results for the ill-conditioned P3WLS zones, outperforming P3WLS itself and 2WLS. Mismatch between simulation and experiment could be explained by noise model incompleteness. This paper took only Gaussian noise model into consideration, however original paper studied also Reference Anchor desynchronization and Shadowing. Complex environment probably has caused difficulties for 2WLS and P3WLS algorithms, meaning QWLS has better robustness under such conditions, these will be studied in future works.

Future work will focus on deriving low-complexity iterative or closed-form approximations to QWLS for real-time implementation, as well as exploring hybrid localization frameworks that dynamically adapt to geometric conditions using α and λ -based criteria. Additionally, the generality of the approach suggests its applicability beyond UWB, to other TDoA-based sensing systems operating in noisy or reflective environments.

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