Differential and Common Mode Noise Waves and Correlation Matrices

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Abstract—**This paper presents an innovative extention of the noise wave definition to mixed mode, differential – and common-mode noise waves which can be used for noise analysis of differential microwave networks. Mixed mode noise waves are used next to define generalized mixed mode noise wave correlation matrices of microwave multiport networks. Presented approach may be used for noise analysis of microwave differential networks with differential ports as well as with conventional single ended ports.**

Keywords—**differential and common mode noise waves, mixed-mode noise wave correlation matrices, generalized multiport mixed-mode noise wave correlation matrices**

I. INTRODUCTION

MANY present day RF and microwave networks are implemented as differential networks. Such networks require appropriate tools for characterization, analysis and design. In 1995, D. Bockelman and W.R. Eisenstadt [1] introduced so-called mixed-mode waves (wave variables) and mixed-mode scattering parameters to extend the classical single-ended wave approach to the differential case. In 2006, A. Ferrero and M. Pirolla [2] introduced so-called generalized mixed-mode scattering matrix which may be used for hybrid networks having some ports differential and some ports single-ended. Such theory may be used for characterization and signal analysis and design of differential RF and microwave networks containing differential amplifiers, baluns, transformers etc.

In this paper we introduce differential- and common- mode noise waves as wave variables which are used for noise analysis of differential microwave networks. Differential-and common-mode noise wave definitions are based on the pseudowave definition presented in [3] by R. Marks and D. Williams. We present the relation of mixed-mode, differential- and common-mode noise waves to the single ended, standard noise waves. We introduce the differential- and common-mode noise wave correlation matrices, discuss their properties and their relation to standard, single-ended noise wave correlation matrices. We introduce also the generalized mixed-mode noise wave correlation matrix for multiport microwave networks with mixed-mode, differential- and common-mode ports as well as with single-ended ports. We derive and discuss the mixed-mode noise wave correlation matrices for passive multiport networks. Finally, we present some application examples

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deriving mixed mode scattering matrices of passive single-ended two-ports, passive single–ended four-ports, baluns and for differential amplifiers.

II. DIFFERENTIAL AND COMMON MODE NOISE WAVES

As for a general n-port network excited by stationary signal we define the single-ended noise voltage and noise current state vector for port j as

$$\mathbf{r}_{\mathrm{n}\,j} \equiv \begin{bmatrix} v_{\mathrm{n}\,j} & i_{\mathrm{n}\,j} \end{bmatrix}^{\mathrm{T}} \tag{1}$$

where $v_{\mathrm{n}\,j}$ and $i_{\mathrm{n}\,j}$ are noise voltage and noise current at the port.

At single-ended port pair j,k of the mixed-mode port set, we define the differential- and common-mode noise voltages and noise currents as

$$v_{\mathrm{nd}\,jk} \equiv v_{\mathrm{n}\,j} - v_{\mathrm{n}\,k} \tag{2}$$

$$i_{\mathrm{nd}\,jk} \equiv \frac{i_{\mathrm{n}\,j} - i_{\mathrm{n}\,k}}{2} \tag{3}$$

$$v_{\mathrm{nc}\,jk} \equiv \frac{v_{\mathrm{n}\,j} + v_{\mathrm{n}\,k}}{2} \tag{4}$$

$$i_{\mathrm{nc}\,jk} \equiv i_{\mathrm{n}\,j} + i_{\mathrm{n}\,k} \tag{5}$$

The state vector containing the mixed-mode noise voltages and noise currents of the port pair j and k is

$$\overset{\circ}{\mathbf{r}}_{\mathrm{n}\,jk} = \begin{bmatrix} v_{\mathrm{nd}\,jk} & i_{\mathrm{nd}\,jk} & v_{\mathrm{nc}\,jk} & i_{\mathrm{nc}\,jk} \end{bmatrix}^{T} \tag{6}$$

Relations (1) through (6) can be presented in matrix notation as

$$\overset{\mathrm{o}}{\mathbf{r}}_{\mathrm{n}\,jk} = \mathbf{T}\,\mathbf{r}_{\mathrm{n}\,jk} \tag{7}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{8}$$

and

$$\mathbf{r}_{\mathrm{n}\,jk} \equiv \begin{bmatrix} \mathbf{r}_{\mathrm{n}\,j} \\ \mathbf{r}_{\mathrm{n}\,k} \end{bmatrix} \tag{9}$$

By analogy to the single-ended port noise pseudowaves defined in [4] as

$$a_{\mathrm{n}\,i} \equiv \frac{\sqrt{R_i}}{2|Z_i|}\left(v_{\mathrm{n}\,i} + i_{\mathrm{n}\,i} Z_i \right) \tag{10}$$

$$b_{\mathrm{n}i} \equiv \frac{\sqrt{R_i}}{2|Z_i|}\left(v_{\mathrm{n}i} - i_{\mathrm{n}i}Z_i\right) \tag{11}$$

where $R_i = \mathrm{Re}\{Z_i\}$, and Z_i is the reference impedance, we define the mixed-mode noise pseudowaves corresponding to a mixed-mode port as

$$a_{\mathrm{nd}jk} \equiv \frac{\sqrt{R_{\mathrm{d}jk}}}{2|Z_{\mathrm{d}jk}|}\left(v_{\mathrm{nd}jk} + i_{\mathrm{nd}jk}Z_{\mathrm{d}jk}\right) \tag{12}$$

$$b_{\mathrm{nd}jk} \equiv \frac{\sqrt{R_{\mathrm{d}jk}}}{2|Z_{\mathrm{d}jk}|}\left(v_{\mathrm{nd}jk} - i_{\mathrm{nd}jk}Z_{\mathrm{d}jk}\right) \tag{13}$$

$$a_{\mathrm{nc}jk} \equiv \frac{\sqrt{R_{\mathrm{c}jk}}}{2|Z_{\mathrm{c}jk}|}\left(v_{\mathrm{nc}jk} + i_{\mathrm{nc}jk}Z_{\mathrm{c}jk}\right) \tag{14}$$

$$b_{\mathrm{nc}jk} \equiv \frac{\sqrt{R_{\mathrm{c}jk}}}{2|Z_{\mathrm{c}jk}|}\left(v_{\mathrm{nc}jk} - i_{\mathrm{nc}jk}Z_{\mathrm{c}jk}\right) \tag{15}$$

where j and k indicate port pair of the mixed-mode port, $v_{\mathrm{nd}jk}$ and $i_{\mathrm{n}\,jk}$ are the differential-mode noise voltage and current defined by (2) and (3), $v_{\mathrm{nc}jk}$ and $i_{\mathrm{nc}jk}$ are the common-mode noise voltage and current defined by (4) and (5), $Z_{\mathrm{c}jk}$, is the reference impedance for the common mode, $Z_{\mathrm{d}jk}$ is the reference impedance for the differential mode, and $R_{\mathrm{c}jk} = \mathrm{Re}\{Z_{\mathrm{c}jk}\}$, and $\mathrm{R}_{\mathrm{d}\,jk} = \mathrm{Re}\{Z_{\mathrm{d}jk}\}$.

Following matrix formalism presented in [1] for the deterministic signal pseudowaves, we can write similar relations for the noise pseudowaves. We have the port state vector in terms of the noise waves defined as

$$\mathbf{w}_{\mathrm{n}\,j} \equiv \begin{bmatrix} a_{\mathrm{n}\,j} \\ b_{\mathrm{n}\,j} \end{bmatrix} = \mathbf{M}_j \begin{bmatrix} v_{\mathrm{n}\,j} \\ i_{\mathrm{n}\,j} \end{bmatrix} = \mathbf{M}_j \mathbf{r}_{\mathrm{n}\,j} \tag{16}$$

where

$$\mathbf{M}_j = \frac{\sqrt{R_j}}{2|Z_j|}\begin{bmatrix} 1 & Z_j \\ 1 & -Z_j \end{bmatrix} \tag{17}$$

By analogy to (16), we can write the mixed-mode noise wave state vector as

$$\overset{\mathrm{o}}{\mathbf{w}}_{\mathrm{n}\,jk} = \begin{bmatrix} a_{\mathrm{nd}\,jk} \\ b_{\mathrm{nd}\,jk} \\ a_{\mathrm{nc}\,jk} \\ b_{\mathrm{nc}\,jk} \end{bmatrix} = \overset{\mathrm{o}}{\mathbf{M}}_{jk} \begin{bmatrix} v_{\mathrm{nd}\,jk} \\ i_{\mathrm{nd}\,jk} \\ v_{\mathrm{nc}\,jk} \\ i_{\mathrm{nc}\,jk} \end{bmatrix} = \overset{\mathrm{o}}{\mathbf{M}}_{jk} \overset{\mathrm{o}}{\mathbf{r}}_{\mathrm{n}\,jk} \tag{18}$$

where

$$\overset{\mathrm{o}}{\mathbf{M}}_{jk} = \begin{bmatrix} \dfrac{\sqrt{R_{\mathrm{d}\,jk}}}{2|Z_{\mathrm{d}\,jk}|} & \dfrac{Z_{\mathrm{d}\,jk}\sqrt{R_{\mathrm{d}\,jk}}}{2|Z_{\mathrm{d}\,jk}|} & 0 & 0 \\[3mm] \dfrac{\sqrt{R_{\mathrm{d}\,jk}}}{2|Z_{\mathrm{d}\,jk}|} & -\dfrac{Z_{\mathrm{d}\,jk}\sqrt{R_{\mathrm{d}\,jk}}}{2|Z_{\mathrm{d}\,jk}|} & 0 & 0 \\[3mm] 0 & 0 & \dfrac{\sqrt{R_{\mathrm{c}\,jk}}}{2|Z_{\mathrm{c}\,jk}|} & \dfrac{Z_{\mathrm{c}\,jk}\sqrt{R_{\mathrm{c}\,jk}}}{2|Z_{\mathrm{c}\,jk}|} \\[3mm] 0 & 0 & \dfrac{\sqrt{R_{\mathrm{c}\,jk}}}{2|Z_{\mathrm{c}\,jk}|} & -\dfrac{Z_{\mathrm{c}\,jk}\sqrt{R_{\mathrm{c}\,jk}}}{2|Z_{\mathrm{c}\,jk}|} \end{bmatrix} \tag{19}$$

We define the overall noise wave state vector for the j-k port pair as

$$\mathbf{w}_{\mathrm{n}\,jk} = \begin{bmatrix} \mathbf{w}_{\mathrm{n}\,j} \\ \mathbf{w}_{\mathrm{n}\,k} \end{bmatrix} \tag{20}$$

and from (16) and (17), we have

$$\mathbf{w}_{\mathrm{n}\,jk} = \mathbf{M}_{jk}\begin{bmatrix} \mathbf{r}_{\mathrm{n}\,j} \\ \mathbf{r}_{\mathrm{n}\,k} \end{bmatrix} = \mathbf{M}_{jk}\,\mathbf{r}_{\mathrm{n}\,jk} = \begin{bmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_k \end{bmatrix}\mathbf{r}_{\mathrm{n}\,jk} \tag{21}$$

Finally, using (16) through (21), we receive the relationship between the standard single-ended noise pseudo-waves and the mixed-mode noise pseudowaves

$$\overset{\mathrm{o}}{\mathbf{w}}_{\mathrm{n}\,jk} = \overset{\mathrm{o}}{\mathbf{M}}_{jk}\,\mathbf{T}\mathbf{M}_{jk}^{-1}\mathbf{w}_{\mathrm{n}\,jk} = \mathbf{X}_{jk}\mathbf{w}_{\mathrm{n}\,jk} \tag{22}$$

where

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$$\mathbf{X}_{jk} = \begin{bmatrix} \dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_j + Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{d}\,jk}||Z_j|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_j - Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{d}\,jk}||Z_j|} & -\dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_k + Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{d}\,jk}||Z_k|} & -\dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_k - Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{d}\,jk}||Z_k|} \\[4mm] \dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_j - Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{d}\,jk}||Z_j|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_j + Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{d}\,jk}||Z_j|} & -\dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_k - Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{d}\,jk}||Z_k|} & -\dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_k + Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{d}\,jk}||Z_k|} \\[4mm] \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_j + 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{c}jk}||Z_j|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_j - 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{c}jk}||Z_j|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_k + 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{c}\,jk}||Z_k|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_k - 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{c}\,jk}||Z_k|} \\[4mm] \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_j - 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{c}\,jk}||Z_j|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_j + 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_j}|Z_{\mathrm{c}\,jk}||Z_j|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_k - 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{c}\,jk}||Z_k|} & \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_k + 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_k}|Z_{\mathrm{c}\,jk}||Z_k|} \end{bmatrix} \tag{23}$$

In (10) through (23), $R_j = \mathrm{Re}\{Z_j\}$, $R_k = \mathrm{Re}\{Z_k\}$, $R_{d\,jk} = \mathrm{Re}\{Z_{d\,jk}\}$ and $R_{c\,jk} = \mathrm{Re}\{Z_{c\,jk}\}$.

The matrix $\mathbf{X}_{jk}$ given by (23) is the same as (16) in [1] which presents relationship between the deterministic sinusoidal signal single-ended pseudowaves and the mixed-mode pseudowaves.

Relations (21) and (23) are valid for the complex reference impedances Z_j, Z_k, $Z_{d\,jk}$ and $Z_{c\,jk}$. If we assume that the single-ended port reference impedances are the same and real: $Z_j = Z_k = Z_R$, and that $Z_{djk} = 2\,Z_R$ and $Z_{cjk} = Z_R/2$, than (23) simplifies to

$$\mathbf{X}_{jk} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad (24)$$

and from (22), we get

$$a_{\mathrm{nd}\,jk} = \frac{1}{\sqrt{2}}\left(a_{\mathrm{n}\,j} - a_{\mathrm{n}\,k}\right) \qquad (25)$$

$$b_{\mathrm{nd}\,jk} = \frac{1}{\sqrt{2}}\left(b_{\mathrm{n}\,j} - b_{\mathrm{n}\,k}\right) \qquad (26)$$

$$a_{\mathrm{nc}\,jk} = \frac{1}{\sqrt{2}}\left(a_{\mathrm{n}\,j} + a_{\mathrm{n}\,k}\right) \qquad (27)$$

$$b_{\mathrm{nc}\,jk} = \frac{1}{\sqrt{2}}\left(b_{\mathrm{n}\,j} + b_{\mathrm{n}\,k}\right) \qquad (28)$$

These relations are the same as (18) in [1] and (3-4) in [2] for the deterministic sinusoidal signal pseudowaves. It is important to notice here that only when reference impedances are real and when they satisfy relations $Z_j = Z_k = Z_R$, $Z_{djk} = 2\,Z_R$ and $Z_{cjk} = Z_R/2$, the ingoing differential-mode noise wave $a_{\mathrm{nd}\,jk}$ and the common-mode noise wave $a_{\mathrm{nc}\,jk}$ depend only on the single-ended ingoing noise waves $a_{\mathrm{n}\,j}$ and $a_{\mathrm{n}\,k}$, and simultaneously the outgoing differential-mode noise wave $b_{\mathrm{nd}jk}$ and common-mode noise wave $b_{\mathrm{nc}jk}$ depend only on the outgoing single-ended noise waves $b_{\mathrm{n}j}$ and $b_{\mathrm{n}k}$. In general case, relations (24) through (28) are not true [1].

From this place of the text in this paper we will omit index "n" at symbols representing noise quantities.

III. GENERALIZED MIXED-MODE NOISE WAVE CORRELATION MATRIX

Fig. 1 illustrates a noisy multiport network having the conventional single-ended ports and the mixed-mode, differential- and common-mode ports.

For this network we can write the matrix equation

$$\overset{\mathrm{o}}{\mathbf{b}}_{\mathrm{n}} = \overset{\mathrm{o}}{\mathbf{S}}\,\overset{\mathrm{o}}{\mathbf{a}}_{\mathrm{n}} + \overset{\mathrm{o}}{\mathbf{c}} \qquad (29)$$
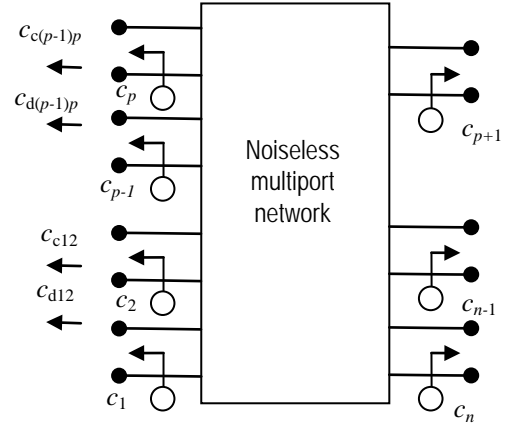
where

Fig. 1. Noisy multiport network with p mixed mode ports and n – p single ended ports.

$$\overset{\mathrm{o}}{\mathbf{a}}_{\mathrm{n}} = \begin{bmatrix} a_{\mathrm{d}12} \\ a_{\mathrm{d}34} \\ . \\ . \\ a_{\mathrm{d}(p-1)p} \\ a_{\mathrm{c}12} \\ a_{\mathrm{c}34} \\ . \\ . \\ a_{\mathrm{c}(p-1)p} \\ a_{p+1} \\ . \\ . \\ a_{n-1} \\ a_n \end{bmatrix} \quad \overset{\mathrm{o}}{\mathbf{b}}_{\mathrm{n}} = \begin{bmatrix} b_{\mathrm{d}12} \\ b_{\mathrm{d}34} \\ . \\ . \\ b_{\mathrm{d}(p-1)p} \\ b_{\mathrm{c}12} \\ b_{\mathrm{c}34} \\ . \\ . \\ b_{\mathrm{c}(p-1)p} \\ b_{p+1} \\ . \\ . \\ b_{n-1} \\ b_n \end{bmatrix} \quad \overset{\mathrm{o}}{\mathbf{c}} = \begin{bmatrix} c_{\mathrm{d}12} \\ c_{\mathrm{d}34} \\ . \\ . \\ c_{\mathrm{d}(p-1)p} \\ c_{\mathrm{n}12} \\ c_{\mathrm{c}34} \\ . \\ . \\ c_{\mathrm{c}(p-1)p} \\ c_{p+1} \\ . \\ . \\ c_{n-1} \\ c_n \end{bmatrix}$$

$$(30)$$

$\overset{\mathrm{o}}{\mathbf{a}}_{\mathrm{n}}$ and $\overset{\mathrm{o}}{\mathbf{b}}_{\mathrm{n}}$ are the generalized mixed-mode noise wave vectors, in which we combine the p mixed-mode noise waves sets with the remaining $(n-p)$ single-ended noise waves, and $\overset{\mathrm{o}}{\mathbf{c}}$ is a vector of p equivalent mixed-mode noise wave sets and $(n-p)$ equivalent single-ended noise waves representing noise generated inside the noisy multiport network.

The matrix $\overset{\mathrm{o}}{\mathbf{S}}$ in (29) is the generalized mixed-mode scattering matrix defined by (26) in [1].

Because the equivalent noise pseudo-wave source representing noise generated in a network is expressed as [3,4]

$$c_i \equiv \frac{\sqrt{R_i}}{2|Z_i|}\left(v_i - i_i Z_i\right) \qquad (31)$$

then for a pair j,k of two ports, forming mixed mode port set, we can write

$$\mathbf{u}_{jk} = \begin{bmatrix} c_j \\ c_k \end{bmatrix} = \mathbf{N}_{jk} \begin{bmatrix} v_j \\ i_j \\ v_k \\ i_k \end{bmatrix} = \mathbf{N}_{jk} \begin{bmatrix} \mathbf{n}_j \\ \mathbf{n}_k \end{bmatrix} = \mathbf{N}_{jk}\mathbf{n}_{jk} \quad (32)$$

where

$$\mathbf{N}_{jk} = \begin{bmatrix} \mathbf{N}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_k \end{bmatrix} \quad (33)$$

$$\mathbf{N}_j = \frac{\sqrt{R_j}}{2|Z_j|}\begin{bmatrix} 1 & -Z_j \end{bmatrix} \quad (34)$$

$$\mathbf{N}_k = \frac{\sqrt{R_k}}{2|Z_k|}\begin{bmatrix} 1 & -Z_k \end{bmatrix} \quad (35)$$

$$\mathbf{n}_{jk} = \begin{bmatrix} v_j & i_j & v_k & i_k \end{bmatrix}^{\mathrm{T}} \quad (36)$$

$\mathbf{0}$ is 1 x 2 zero matrix, Z_i, Z_j, Z_k are the reference impedances, $R_i = \mathrm{Re}\{Z_i\}$, $R_j = \mathrm{Re}\{Z_j\}$ and $R_k = \mathrm{Re}\{Z_k\}$.

Using the definitions of the differential-mode and common-mode noise voltages and noise currents, given by (2) through (5), we introduce the differential- and common-mode equivalent noise pseudo-wave sources representing noise generated in the network

$$c_{\mathrm{d}\,jk} = \frac{\sqrt{R_{\mathrm{d}\,jk}}}{2|Z_{\mathrm{d}\,jk}|}\left(v_{\mathrm{d}\,jk} - i_{\mathrm{d}\,jk}Z_{\mathrm{d}\,jk}\right) \quad (37)$$

$$c_{\mathrm{c}\,jk} = \frac{\sqrt{R_{\mathrm{c}\,jk}}}{2|Z_{\mathrm{c}\,jk}|}\left(v_{\mathrm{c}\,jk} - i_{\mathrm{c}\,jk}Z_{\mathrm{c}\,jk}\right) \quad (38)$$

where j and k indicate port pair of the mixed-mode port, $Z_{\mathrm{c}\,jk}$, - the reference impedance for the common mode, $Z_{\mathrm{d}jk}$ - the reference impedance for differential mode and $R_{\mathrm{c}jk} = \mathrm{Re}\{Z_{\mathrm{c}jk}\}$ and $R_{\mathrm{d}jk} = \mathrm{Re}\{Z_{\mathrm{d}jk}\}$.

From (31) through (38) the port state vector of the mixed-mode noise waves at the mixed-mode port j,k can be written as

$$\overset{\mathrm{o}}{\mathbf{u}}_{jk} \equiv \begin{bmatrix} c_{\mathrm{d}\,jk} \\ c_{\mathrm{c}\,jk} \end{bmatrix} = \overset{\mathrm{o}}{\mathbf{N}}_{jk}\,\overset{\mathrm{o}}{\mathbf{n}}_{jk} \quad (39)$$

where

$$\overset{\mathrm{o}}{\mathbf{N}}_{jk} = \begin{bmatrix} \mathbf{N}_{djk} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{cjk} \end{bmatrix} \quad (40)$$

$$\mathbf{N}_{djk} = \frac{\sqrt{R_{djk}}}{2|Z_{djk}|}\begin{bmatrix} 1 & -Z_{djk} \end{bmatrix} \quad (41)$$

$$\mathbf{N}_{cjk} = \frac{\sqrt{R_{cjk}}}{2|Z_{cjk}|}\begin{bmatrix} 1 & -Z_{cjk} \end{bmatrix} \quad (42)$$

$$\overset{\mathrm{o}}{\mathbf{n}}_{jk} = \begin{bmatrix} v_{djk} & i_{djk} & v_{cjk} & i_{cjk} \end{bmatrix}^{\mathrm{T}} \quad (43)$$

and $\mathbf{0}$ is 1 x 2 zero matrix.

Using (8), (33) and (39) through (43), we can write

$$\overset{\mathrm{o}}{\mathbf{u}}_{jk} \equiv \begin{bmatrix} c_{djk} \\ c_{cjk} \end{bmatrix} = \overset{\mathrm{o}}{\mathbf{N}}_{jk}\,\overset{\mathrm{o}}{\mathbf{n}}_{jk} = \overset{\mathrm{o}}{\mathbf{N}}_{jk}\,\mathbf{T}\mathbf{n}_{jk} = \overset{\mathrm{o}}{\mathbf{N}}_{jk}\,\mathbf{T}\mathbf{N}_{jk}^{-1}\mathbf{u}_{jk} \quad (44)$$

or

$$\overset{\mathrm{o}}{\mathbf{u}}_{jk} \equiv \mathbf{Y}_{jk}\begin{bmatrix} c_j \\ c_k \end{bmatrix} = \mathbf{Y}_{jk}\mathbf{u}_{jk} \quad (45)$$

where

$$\mathbf{Y}_{jk} = \overset{\mathrm{o}}{\mathbf{N}}_{jk}\,\mathbf{T}\mathbf{N}_{jk}^{-1} =$$

$$= \begin{bmatrix} \dfrac{|Z_j|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_j + Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_j}\,|Z_{\mathrm{d}\,jk}||Z_j|} & -\dfrac{|Z_k|\sqrt{R_{\mathrm{d}\,jk}}\left(2Z_k + Z_{\mathrm{d}\,jk}\right)}{4\sqrt{R_k}\,|Z_{\mathrm{d}\,jk}||Z_k|} \\ \dfrac{|Z_j|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_j + 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_j}\,|Z_{\mathrm{c}\,jk}||Z_j|} & \dfrac{|Z_k|\sqrt{R_{\mathrm{c}\,jk}}\left(Z_k + 2Z_{\mathrm{c}\,jk}\right)}{4\sqrt{R_k}\,|Z_{\mathrm{c}\,jk}||Z_k|} \end{bmatrix}$$

$$(46)$$

Equation (45) is the generalized relation between the single-ended equivalent noise wave sources and the mixed-mode equivalent noise wave sources representing noise generated in a multiport. Adopting a real and the same reference impedance for both ports $Z_j = Z_k = Z_{\mathrm{R}}$ and assuming $Z_{\mathrm{d}jk} = 2\,Z_{\mathrm{R}}$ and $Z_{\mathrm{c}jk} = Z_{\mathrm{R}}/2$, (46) simplifies to

$$\mathbf{Y}_{jk} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad (47)$$

which, substituted in (45), gives

$$c_{djk} = \frac{c_j - c_k}{\sqrt{2}} \quad (48)$$

$$c_{cjk} = \frac{c_j + c_k}{\sqrt{2}} \quad (49)$$

Generalized relation for all ports of the mixed-mode noisy network will have the form

$$\overset{\mathrm{o}}{\mathbf{u}} = \mathbf{\Psi}\mathbf{u} \quad (50)$$

where

$$\overset{\mathrm{o}}{\mathbf{u}} = \begin{bmatrix} \overset{\mathrm{o}}{\mathbf{u}}_{12} \\ \overset{\mathrm{o}}{\mathbf{u}}_{34} \\ \vdots \\ \overset{\mathrm{o}}{\mathbf{u}}_{(p-1)p} \\ c_{p+1} \\ c_{p+2} \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_{12} \\ \mathbf{u}_{34} \\ \vdots \\ \mathbf{u}_{(p-1)p} \\ c_{p+1} \\ c_{p+2} \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} \quad (51)$$

$\overset{\mathrm{o}}{\mathbf{u}}$ is the column vector of the equivalent noise wave sources at the mixed-mode ports and at the single-ended ports of a general differential multiport, $\mathbf{u}$ is the column vector of the equivalent noise wave sources at the single-ended ports of a multiport, and

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{Y}_{12} & \mathbf{0} & .. & & .. & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{34} & .. & & .. & \vdots \\ \vdots & . & . & & .. & ; \\ \vdots & . & . & \mathbf{Y}_{(p-1)p} & & \mathbf{0} \\ \mathbf{0} & . & . & & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (52)$$

is the coefficient matrix in which submatrices $\mathbf{Y}_{ij}$ are given by (46). In (52) $\mathbf{I}$ is the $(n - p)$ x $(n - p)$ identity matrix.

To determine the mixed-mode noise wave correlation matrix compatible with the generalized mixed-mode scattering matrix, elements of column vector $\overset{o}{\mathbf{u}}$, given by (51), have to reordered to the form

$$\overset{o}{\mathbf{c}} = \begin{bmatrix} c_{d12} \\ c_{d34} \\ .. \\ c_{d(p-1)p} \\ c_{c12} \\ c_{c34} \\ .. \\ c_{c(p-1)p} \\ c_{p+1} \\ c_{p+2} \\ .. \\ c_n \end{bmatrix} \tag{53}$$

This ordering may be done using a relation

$$\overset{o}{\mathbf{c}} = \begin{bmatrix} \mathbf{P}_d \\ \mathbf{P}_c \\ \mathbf{P}_I \end{bmatrix} \overset{o}{\mathbf{u}} = \mathbf{P} \overset{o}{\mathbf{u}} = \mathbf{P}\mathbf{\Psi}\mathbf{u} \tag{54}$$

in which $\mathbf{P}$ is the n x n permutation matrix. Submatrices $\mathbf{P}_d$ and $\mathbf{P}_c$ are of size $p/2$ x n, while submatrix $\mathbf{P}_I$ is of size $(n − p)$ x n. These matrices have all elements null apart a single 1 in each row:

$$P_d(l, 2l − 1) = 1; \quad l = 1, 2, …, p/2; \qquad P_d = 0, \text{ elsewhere} \tag{55}$$

$$P_c(l, 2l) = 1, \quad l = 1, 2, …, p/2; \qquad P_c = 0, \text{ elsewhere} \tag{56}$$

$$P_I(l,l) = 1, \quad l = 1, 2, 3, … , (n-p); \qquad P_I = 0, \text{ elsewhere} \tag{57}$$

Using (53) and (54) the generalized mixed-mode noise wave correlation matrix of a multiport can be derived as

$$\overset{o}{\mathbf{C}}_S = \overline{\overset{o}{\mathbf{c}}\,\overset{o}{\mathbf{c}}^+} = \mathbf{P}\mathbf{\Psi}\overline{\mathbf{u}\mathbf{u}^+}\mathbf{\Psi}^+\mathbf{P}^+ = \mathbf{P}\mathbf{\Psi}\mathbf{C}_S\mathbf{\Psi}^+\mathbf{P}^+ \tag{58}$$

In (58) $\mathbf{C}_S = \overline{\mathbf{u}\mathbf{u}^+}$ is the standard single-ended noise wave correlation matrix of a multiport discussed in [4]. Equation (58) represents the generalized form of the single-ended noise wave correlation matrix to the generalized mixed-mode noise wave correlation matrix transformation.

As all noise correlation matrices, the generalized mixed-mode noise wave correlation matrices $\overset{o}{\mathbf{C}}_S$ are Hermitian matrices, what means that $\overset{o}{\mathbf{C}}_S = \overset{o}{\mathbf{C}}_S^+ = \overset{o}{\mathbf{C}}_S^{*T}$.

Because the vector $\overset{o}{\mathbf{c}}$ may be written as

$$\overset{o}{\mathbf{c}} = \begin{bmatrix} \mathbf{c}_d \\ \mathbf{c}_c \\ \mathbf{c}_s \end{bmatrix} \tag{59}$$

the generalized mixed-mode correlation matrix $\overset{o}{\mathbf{C}}_S$ may be partitioned into nine submatrices

$$\overset{o}{\mathbf{C}}_S = \begin{bmatrix} \overset{o}{\mathbf{C}}_{dd} & \overset{o}{\mathbf{C}}_{dc} & \overset{o}{\mathbf{C}}_{ds} \\ \overset{o}{\mathbf{C}}_{cd} & \overset{o}{\mathbf{C}}_{cc} & \overset{o}{\mathbf{C}}_{cs} \\ \overset{o}{\mathbf{C}}_{sd} & \overset{o}{\mathbf{C}}_{sc} & \mathbf{C}_S \end{bmatrix} \tag{60}$$

highlighting the correlation and cross-correlation between the differential-, common-, and single-ended mode equivalent noise wave sources representing noise generated in a mixed-mode multiport. In (59) and (60), the "d", "c" and "s" subscripts at submatrices refer to the differential, common and single-ended mode, respectively.

If the network is a single-ended four-port and its ports 1 and 2 are a pair forming the mixed-mode port, and similarly ports 3 and 4 are a pair forming the second mixed-mode port, and it is assumed that $Z_{d12} = Z_{d34} = 2 Z_R$ and $Z_{c12} = Z_{c34} = Z_R/2$, where Z_R is the real reference impedance, common to all four single-ended ports of the network, then

$$\overset{o}{\mathbf{C}}_S = \begin{bmatrix} \overline{|c_{d12}|^2} & \overline{c_{d12}c_{d34}^*} & \overline{c_{d12}c_{c12}^*} & \overline{c_{d12}c_{c34}^*} \\ \overline{c_{d21}c_{d1}^*} & \overline{|c_{d34}|^2} & \overline{c_{d34}c_{c12}^*} & \overline{c_{d34}c_{c34}^*} \\ \overline{c_{c12}c_{d12}^*} & \overline{c_{c12}c_{d34}^*} & \overline{|c_{c12}|^2} & \overline{c_{c12}c_{c34}^*} \\ \overline{c_{c34}c_{d12}^*} & \overline{c_{c34}c_{d34}^*} & \overline{c_{c34}c_{c12}^*} & \overline{|c_{c34}|^2} \end{bmatrix}$$

$$\tag{63}$$

$$= \frac{1}{2} \begin{bmatrix} C_{11} + C_{22} - 2\operatorname{Re}\{C_{12}\} & C_{13} - C_{14} - C_{23} + C_{24} & C_{11} - C_{22} + j2\operatorname{Im}\{C_{12}\} & C_{13} + C_{14} - C_{23} - C_{24} \\ C_{31} - C_{41} - C_{32} + C_{42} & C_{33} + C_{44} - 2\operatorname{Re}\{C_{34}\} & C_{31} + C_{32} - C_{41} - C_{42} & C_{33} - C_{44} + j2\operatorname{Im}\{C_{34}\} \\ C_{11} - C_{22} - j2\operatorname{Im}\{C_{12}\} & C_{13} + C_{23} - C_{14} - C_{24} & C_{11} + C_{22} + 2\operatorname{Re}\{C_{12}\} & C_{13} + C_{14} + C_{23} + C_{24} \\ C_{31} - C_{32} + C_{41} - C_{42} & C_{33} - C_{44} - j2\operatorname{Im}\{C_{34}\} & C_{31} - C_{32} - C_{41} + C_{42} & C_{33} + C_{44} + 2\operatorname{Re}\{C_{34}\} \end{bmatrix}$$

$$\overset{\text{o}}{\mathbf{C}}_S = \begin{bmatrix} \overline{|c_{d12}|^2} & \overline{c_{d12}c_{c12}^*} & \overline{c_{d12}c_3^*} \\ \overline{c_{c12}c_{d12}^*} & \overline{|c_{c12}|^2} & \overline{c_{c12}c_3^*} \\ \overline{c_3c_{d12}^*} & \overline{c_3c_{c12}^*} & \overline{|c_3|^2} \end{bmatrix}$$

(65)

$$= \frac{1}{2}\begin{bmatrix} C_{11}+C_{22}-2\text{Re}\{C_{12}\} & C_{11}-C_{22}+j2\text{Im}\{C_{12}\} & \sqrt{2}(C_{13}-C_{23}) \\ C_{11}-C_{22}-j2\text{Im}\{C_{12}\} & C_{11}+C_{22}+2\text{Re}\{C_{12}\} & \sqrt{2}(C_{13}+C_{23}) \\ \sqrt{2}(C_{13}^*-C_{23}^*) & \sqrt{2}(C_{13}^*+C_{23}^*) & 2C_{33} \end{bmatrix}$$

$$\mathbf{P\Psi} = \mathbf{M} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(61)

and (58) becomes

$$\overset{\text{o}}{\mathbf{C}}_S = \mathbf{M}\mathbf{C}_S\mathbf{M}^+$$

(62)

Using (61) and (62), we get the generalized mixed-mode noise wave correlation matrix $\overset{\text{o}}{\mathbf{C}}_S$ in terms of the elements of standard, single-ended noise wave correlation matrix $\mathbf{C}_S$ as (63) shown at the bottom of the previous page.

For a single-ended three-port with ports 1 and 2 forming the mixed-mode port, and assuming that $Z_{d12} = 2 Z_R$ and $Z_{c12} = Z_R/2$, where Z_R is the real reference impedance of the single-ended ports, we have

$$\mathbf{M} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

(64)

The generalized mixed-mode noise wave correlation matrix $\overset{\text{o}}{\mathbf{C}}_S$ of such network, expressed in terms of standard, single-ended noise wave correlation matrix $\mathbf{C}_S$ elements, is given by (65) presented at the top of this page.

For a single-ended two-port network considered as a mixed-mode one-port

$$\mathbf{M} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(66)

and the mixed-mode noise wave correlation matrix $\overset{\text{o}}{\mathbf{C}}_S$ of such network, expressed in terms of the standard, single-ended noise wave correlation matrix $\mathbf{C}_S$ elements, is

$$\overset{\text{o}}{\mathbf{C}}_S = \begin{bmatrix} \overset{\text{o}}{C}_{11} & \overset{\text{o}}{C}_{12} \\ \overset{\text{o}}{C}_{12}^* & \overset{\text{o}}{C}_{22} \end{bmatrix} = \begin{bmatrix} C_{dd} & C_{dc} \\ C_{cd} & C_{cc} \end{bmatrix} = \begin{bmatrix} \overline{|c_{d12}|^2} & \overline{c_{d12}c_{c12}^*} \\ \overline{c_{c12}c_{d12}^*} & \overline{|c_{c12}|^2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} C_{11}+C_{22}-2\text{Re}\{C_{12}\} & C_{11}-C_{22}+j2\text{Im}\{C_{12}\} \\ C_{11}-C_{22}-j2\text{Im}\{C_{12}\} & C_{11}+C_{22}+2\text{Re}\{C_{12}\} \end{bmatrix}$$

(67)

$$= \frac{1}{2}\begin{bmatrix} \overline{|c_1|^2}+\overline{|c_2|^2}-2\text{Re}\{\overline{c_1c_2^*}\} & \overline{|c_1|^2}-\overline{|c_2|^2}+2j\text{Im}\{\overline{c_1c_2^*}\} \\ \overline{|c_1|^2}-\overline{|c_2|^2}-2j\text{Im}\{\overline{c_1c_2^*}\} & \overline{|c_1|^2}+\overline{|c_2|^2}+2\text{Re}\{\overline{c_1c_2^*}\} \end{bmatrix}$$

IV. MIXED-MODE NOISE WAVE CORRELATION MATRICES OF PASSIVE NETWORKS

As it was discussed in [5], the noise wave correlation matrix of a passive single-ended multiport with real port reference impedances Z_R, equals

$$\mathbf{C}_S = kT(\mathbf{I} - \mathbf{S}\mathbf{S}^+)$$

(68)

Using (62) and (68), and assuming that for all the mixed-mode ports reference impedances are real, and that $Z_{d\,jk} = 2 Z_R$, and $Z_{c\,jk} = Z_R/2$, it can be proved that for passive mixed-mode multiport networks described by their generalized mixed-mode scattering matrices $\overset{\text{o}}{\mathbf{S}}$, the generalized mixed-mode noise wave correlation matrix is expressed by the similar to (68) relation

$$\overset{\text{o}}{\mathbf{C}}_S = kT\left(\mathbf{I} - \overset{\text{o}}{\mathbf{S}}\overset{\text{o}}{\mathbf{S}}^+\right)$$

(69)

where + indicates the Hermitian conjugate.

A. Mixed-mode noise wave correlation matrix of a passive two-port network

Measurement of differential networks noise properties is still a complicated problem at microwave frequencies. A method based on a hot/cold differential load used as a differential noise source has been presented in [6].

Figure 2 illustrates a passive single-ended two-port network considered as a differential load with differential-mode and common mode noise wave sources at its port.
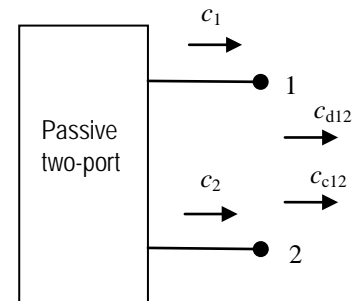

Fig. 2 Passive single-ended two-port considered as a mixed-mode one-port network and a source of noise waves

Using (68), the wave correlation matrix of a single-ended two-port network can be written as

$$\mathbf{C}_{\mathrm{S}} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix} = \begin{bmatrix} \overline{|c_1|^2} & \overline{c_1 c_2^*} \\ \overline{c_1^* c_2} & \overline{|c_2|^2} \end{bmatrix}$$

$$= kT \begin{bmatrix} 1 - |S_{11}|^2 - |S_{12}|^2 & -\left(S_{11}S_{21}^* + S_{12}S_{22}^*\right) \\ -\left(S_{11}^*S_{21} + S_{12}^*S_{22}\right) & 1 - |S_{22}|^2 - |S_{21}|^2 \end{bmatrix}$$

$$(70)$$

From (68), it can be seen that correlation between equivalent noise waves c_1 and c_2 disappears when there is no coupling between ports 1 and 2. This is the obvious conclusion. Equivalent noise waves c_1 and c_2 are also not correlated when a passive two-port network is matched at its port, that is, when $S_{11} = S_{22} = 0$. This result can be explained by examining, how noise power generated in a passive network is delivered to the reflectionless terminations in thermodynamic equilibrium. These considerations can be found in [5].

Now, using (62), (66) and (70), we find the mixed-mode noise wave correlation matrix $\overset{\circ}{\mathbf{C}}_{\mathrm{S}}$ of a passive two-port as

$$\overset{\circ}{C}_{11} = C_{\mathrm{dd}} = \overline{|c_{\mathrm{d12}}|^2}$$
$$= \frac{kT}{2}\left[2 - |S_{11}|^2 - |S_{12}|^2 - |S_{21}|^2 - |S_{22}|^2 + 2\mathrm{Re}\left\{S_{11}S_{21}^* + S_{12}S_{22}^*\right\}\right]$$
$$(71)$$

$$\overset{\circ}{C}_{22} = C_{\mathrm{cc}} = \overline{|c_{\mathrm{c12}}|^2}$$
$$= \frac{kT}{2}\left[2 - |S_{11}|^2 - |S_{12}|^2 - |S_{21}|^2 - |S_{22}|^2 - 2\mathrm{Re}\left\{S_{11}S_{21}^* + S_{12}S_{22}^*\right\}\right]$$
$$(72)$$

$$\overset{\circ}{C}_{12} = C_{\mathrm{dc}} = \overline{c_{\mathrm{d12}}c_{\mathrm{c12}}^*}$$
$$= \frac{kT}{2}\left[-|S_{11}|^2 - |S_{12}|^2 + |S_{21}|^2 + |S_{22}|^2 - 2j\mathrm{Im}\left\{S_{11}S_{21}^* + S_{12}S_{22}^*\right\}\right]$$
$$(73)$$

where k is the Boltzmann's constant and T is the physical temperature of the network.

Considering a passive two-port as a source of the mixed-mode noise waves c_{d12} and c_{c12}, we see from (71) through (73), that the noise temperatures $T_{\mathrm{d}} = \overline{|c_{\mathrm{d12}}|^2}/k$ and $T_{\mathrm{c}} = \overline{|c_{\mathrm{c12}}|^2}/k$, for the differential-mode and common-mode noise waves generated by a single-ended two-port are different.

For a symmetrical passive two-port, with $S_{11} = S_{22}$ and $S_{12} = S_{21}$, we have

$$\overset{\circ}{C}_{11} = C_{\mathrm{dd}} = \overline{|c_{\mathrm{d12}}|^2} = kT\left[1 - |S_{11}|^2 - |S_{12}|^2 + 2\mathrm{Re}\left\{S_{11}S_{12}^*\right\}\right] \quad (74)$$

$$\overset{\circ}{C}_{22} = C_{\mathrm{cc}} = \overline{|c_{\mathrm{c12}}|^2} = kT\left[1 - |S_{11}|^2 - |S_{12}|^2 - 2\mathrm{Re}\left\{S_{11}S_{12}^*\right\}\right] \quad (75)$$

$$\overset{\circ}{C}_{12} = C_{\mathrm{dc}} = \overline{c_{\mathrm{d12}}c_{\mathrm{c12}}^*} = 0 \quad (76)$$

From (74) through (76), we see that the common- and differential-mode noise waves are uncorrelated but the noise temperatures of the common- and differential-mode noise waves are still different. When ports of a two-port network are matched, $S_{11} = S_{22} = 0$, the noise temperatures T_{d} and T_{c} are the same and equal $T\left[1 - |S_{12}|^2\right]$.

B. Mixed-Mode Noise Wave Correlation Matrix of a Balun

It is a common practice to use two baluns or two hybrid couplers at the input and at the output of a differential device under test (DDUT) in order to embed the differential DUT into a single-ended measurement setup [6-8]. Signal as well as noise properties of the DDUT then have to be de-embedded from the results obtained in a single-ended measurement environment.

Figure 3 shows a balun with the equivalent noise wave sources at its ports.
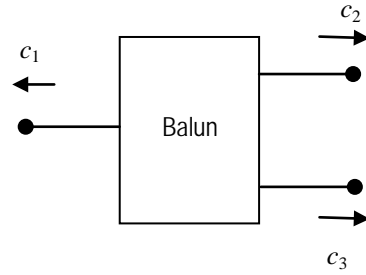

Fig. 3 A balun with the equivalent noise wave sources at its ports. Pair 2-3 of the single ended ports creates a mixed-mode port.

As it is a passive network, its standard noise wave correlation matrix is given as

$$\mathbf{C}_{\mathrm{S}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12}^* & C_{22} & C_{23} \\ C_{13}^* & C_{23}^* & C_{33} \end{bmatrix} = \begin{bmatrix} \overline{|c_1|^2} & \overline{c_1 c_2^*} & \overline{c_1 c_3^*} \\ \overline{c_1^* c_2} & \overline{|c_2|^2} & \overline{c_2 c_3^*} \\ \overline{c_1^* c_3} & \overline{c_2^* c_3} & \overline{|c_3|^2} \end{bmatrix} = kT\left(\mathbf{I} - \mathbf{S}\mathbf{S}^+\right) \quad (77)$$

where $\mathbf{S}$ is the standard, single-ended scattering matrix of a balun.

Using (77), we get

$$\mathbf{C}_{\mathrm{S}} = kT \begin{bmatrix} 1 - |S_{11}|^2 - |S_{12}|^2 - |S_{13}|^2 & -(S_{11}S_{21}^* + S_{12}S_{22}^* + S_{13}S_{23}^*) & -(S_{11}S_{31}^* + S_{12}S_{32}^* + S_{13}S_{33}^*) \\ -(S_{11}S_{21}^* + S_{12}S_{22}^* + S_{13}S_{23}^*)^* & 1 - |S_{21}|^2 - |S_{22}|^2 - |S_{23}|^2 & -(S_{21}S_{31}^* + S_{22}S_{32}^* + S_{23}S_{33}^*) \\ -(S_{11}S_{31}^* + S_{12}S_{32}^* + S_{13}S_{33}^*)^* & -(S_{21}S_{31}^* + S_{22}S_{32}^* + S_{23}S_{33}^*)^* & 1 - |S_{31}|^2 - |S_{32}|^2 - |S_{33}|^2 \end{bmatrix} \quad (78)$$

$$\overset{o}{\mathbf{C}}_{S} = \frac{1}{2} \begin{bmatrix} 2C_{11} & \sqrt{2}\left(C_{12}-C_{13}\right) & \sqrt{2}\left(C_{12}+C_{13}\right) \\ \sqrt{2}\left(C_{12}^{*}-C_{13}^{*}\right) & C_{22}+C_{33}-2\mathrm{Re}\{C_{23}\} & C_{22}-C_{33}+j2\mathrm{Im}\{C_{23}\} \\ \sqrt{2}\left(C_{12}^{*}+C_{13}^{*}\right) & C_{22}-C_{33}+j2\mathrm{Im}\{C_{23}\} & C_{22}+C_{33}+2\mathrm{Re}\{C_{23}\} \end{bmatrix} \tag{82}$$

Assuming that the balun is matched at all ports, that is $S_{11} = S_{22} = S_{33} = 0$, from (78)

$$\mathbf{C}_{S} = kT \begin{bmatrix} 1-\left|S_{12}\right|^{2}-\left|S_{13}\right|^{2} & -S_{13}S_{23}^{*} & -S_{12}S_{32}^{*} \\ -S_{13}^{*}S_{23} & 1-\left|S_{21}\right|^{2}-\left|S_{23}\right|^{2} & -S_{21}S_{31}^{*} \\ -S_{21}^{*}S_{32} & -S_{21}^{*}S_{31} & 1-\left|S_{31}\right|^{2}-\left|S_{32}\right|^{2} \end{bmatrix} \tag{79}$$

Considering the pair of ports 2 and 3 as a mixed-mode port with the equivalent differential- and common-mode noise waves, we can construct and derive generalized mixed-mode noise wave correlation matrix. It is

$$\overset{o}{\mathbf{C}}_{S} = \overline{\begin{bmatrix} c_{1} \\ c_{d23} \\ c_{c23} \end{bmatrix} \begin{bmatrix} c_{1} \\ c_{d23} \\ c_{c23} \end{bmatrix}^{+}} = \begin{bmatrix} \overline{\left|c_{1}\right|^{2}} & \overline{c_{1}c_{d23}^{*}} & \overline{c_{1}c_{c23}^{*}} \\ \overline{c_{d23}c_{1}^{*}} & \overline{\left|c_{d23}\right|^{2}} & \overline{c_{d23}c_{c23}^{*}} \\ \overline{c_{c23}c_{1}^{*}} & \overline{c_{c23}c_{d23}^{*}} & \overline{\left|c_{c23}\right|^{2}} \end{bmatrix} \tag{80}$$

Using (62), with

$$\mathbf{M} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \tag{81}$$

we receive (82), shown at the top of this page. In (82) C_{ij}, $i, j =$ 1,2,3 are the elements of standard noise wave correlation matrix $\mathbf{C}_{S}$ given by (78) or (79).

C. Mixed-Mode Noise Wave Correlation Matrix of Passive Four-Port Networks

Symmetrical single-ended four-port networks, presented in Fig. 4, are commonly used as a mixed mode two-ports in differential networks. For example, two symmetrical coupled transmission lines represent such network.
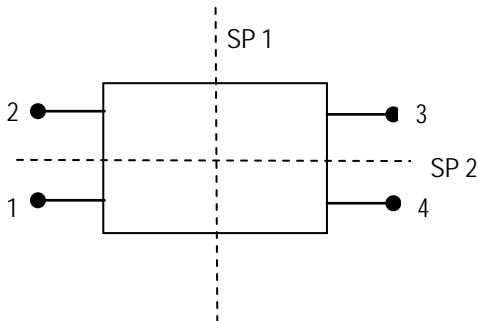

Fig. 4. Symmetrical four-port network with symmetry planes SP1 and SP2.

Single-ended scattering matrix of such networks is

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{14} & S_{13} \\ S_{13} & S_{14} & S_{11} & S_{13} \\ S_{14} & S_{13} & S_{13} & S_{11} \end{bmatrix} \tag{83}$$

Using (61), (62), (68) and (83), we receive

$$\overset{o}{\mathbf{C}}_{S} = \begin{bmatrix} \overset{o}{\mathbf{C}}_{dd} & \overset{o}{\mathbf{C}}_{dc} \\ \overset{o}{\mathbf{C}}_{cd} & \overset{o}{\mathbf{C}}_{cc} \end{bmatrix} = kT \begin{bmatrix} \overset{o}{C}_{11} & \overset{o}{C}_{12} & 0 & 0 \\ \overset{o}{C}_{12}^{*} & \overset{o}{C}_{11} & 0 & 0 \\ 0 & 0 & \overset{o}{C}_{33} & \overset{o}{C}_{34} \\ 0 & 0 & \overset{o}{C}_{34}^{*} & \overset{o}{C}_{33} \end{bmatrix} \tag{84}$$

where

$$\overset{o}{C}_{11} = 1+\mathrm{Re}\{S_{11}S_{12}^{*}+S_{13}S_{14}^{*}\}-\left|S_{11}\right|^{2}-\left|S_{12}\right|^{2}-\left|S_{13}\right|^{2}-\left|S_{14}\right|^{2} \tag{85}$$

$$\overset{o}{C}_{12} = \mathrm{Re}\{S_{11}S_{14}^{*}+S_{12}S_{13}^{*}\}-\mathrm{Re}\{S_{11}S_{13}^{*}+S_{12}S_{14}^{*}\} \tag{86}$$

$$\overset{o}{C}_{33} = 1-\mathrm{Re}\{S_{11}S_{12}^{*}+S_{13}S_{14}^{*}\}-\left|S_{11}\right|^{2}-\left|S_{12}\right|^{2}-\left|S_{13}\right|^{2}-\left|S_{14}\right|^{2} \tag{87}$$

$$\overset{o}{C}_{34} = -\mathrm{Re}\{S_{11}S_{14}^{*}+S_{12}S_{13}^{*}\}-\mathrm{Re}\{S_{11}S_{13}^{*}+S_{12}S_{14}^{*}\} \tag{88}$$

In a case of two identical uncoupled lines ($S_{12} = S_{21} = S_{34} = S_{43} = 0$ and $S_{13} = S_{31} = S_{24} = S_{42} = 0$), relations (85) through (88) simplify to

$$\overset{o}{C}_{11} = 1-\left|S_{11}\right|^{2}-\left|S_{14}\right|^{2} \tag{89}$$

$$\overset{o}{C}_{12} = \mathrm{Re}\{S_{11}S_{14}^{*}\} \tag{90}$$

$$\overset{o}{C}_{33} = 1-\left|S_{11}\right|^{2}-\left|S_{14}\right|^{2} \tag{91}$$

$$\overset{o}{C}_{34} = -\mathrm{Re}\{S_{11}S_{14}^{*}\} \tag{92}$$

V. MIXED-MODE NOISE WAVE CORRELATION MATRICES OF ACTIVE DIFFERENTIAL NETWORKS

Figure 5 illustrates a pair of transistors which form a differential network with two mixed-mode ports.
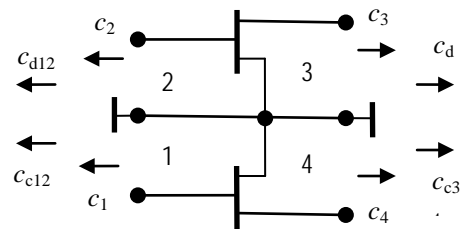

Fig. 5 Differential pair of transistors.

Because the noise generated in both transistors is not correlated, the single-ended noise wave correlation matrix of this network is

$$\mathbf{C}_S = \begin{bmatrix} C_{11} & 0 & 0 & C_{14} \\ 0 & C_{22} & C_{23} & 0 \\ 0 & C_{23}^* & C_{33} & 0 \\ C_{14}^* & 0 & 0 & C_{44} \end{bmatrix} \quad (93)$$

Using (62) and (93), we derive the mixed-mode noise wave correlation matrix as

$$\overset{o}{\mathbf{C}}_S = \begin{bmatrix} \overset{o}{\mathbf{C}}_{dd} & \overset{o}{\mathbf{C}}_{dc} \\ \overset{o}{\mathbf{C}}_{cd} & \overset{o}{\mathbf{C}}_{cc} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} C_{11}+C_{22} & -(C_{14}+C_{23}) & C_{11}-C_{22} & C_{14}-C_{23} \\ -(C_{14}^*+C_{23}^*) & C_{33}+C_{44} & C_{23}^*-C_{14}^* & C_{33}-C_{44} \\ C_{11}-C_{22} & C_{23}-C_{14} & C_{11}+C_{22} & C_{14}+C_{23} \\ C_{14}^*-C_{23}^* & C_{33}-C_{44} & C_{14}^*+C_{23}^* & C_{33}+C_{44} \end{bmatrix} \quad (94)$$

Finally, if we assume now that the transistors used in the network are identical, what means that $C_{11} = C_{22}$, $C_{33} = C_{44}$ and $C_{23} = C_{14}$, (94) simplifies to

$$\overset{o}{\mathbf{C}}_S = \begin{bmatrix} \overset{o}{\mathbf{C}}_{dd} & \overset{o}{\mathbf{C}}_{dc} \\ \overset{o}{\mathbf{C}}_{cd} & \overset{o}{\mathbf{C}}_{cc} \end{bmatrix} = \begin{bmatrix} C_{11} & -C_{14} & 0 & 0 \\ -C_{14}^* & C_{44} & 0 & 0 \\ 0 & 0 & C_{11} & C_{14} \\ 0 & 0 & C_{14}^* & C_{44} \end{bmatrix} \quad (95)$$

In (93) through (95), C_{ij} are the elements of the standard single-ended noise wave correlation matrices of active two-ports (transistors) forming the differential network.

APPENDIX

In (44) and (46) $\mathbf{N}_{jk}^{-1}$ is the one-sided inverse, the right inverse of $\mathbf{N}_{jk}$, what means that $\mathbf{N}_{jk}\mathbf{N}_{jk}^{-1} = \mathbf{I}$, but $\mathbf{N}_{jk}^{-1}\mathbf{N}_{jk} \neq \mathbf{I}$.

$$\mathbf{N}_{jk}^{-1} = \begin{bmatrix} \dfrac{|Z_j|}{\sqrt{R_j}} & 0 \\ -\dfrac{|Z_j|}{\sqrt{R_j}Z_j^*} & 0 \\ 0 & \dfrac{|Z_k|}{\sqrt{R_k}} \\ 0 & -\dfrac{|Z_k|}{\sqrt{R_k}Z_k^*} \end{bmatrix} \quad (A1)$$

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