INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2014, VOL. 60, NO. 4, PP. 331–340
Manuscript received October 3, 2014; revised December 2014. DOI: 10.2478/eletel-2014-0044

Prefiltering in Wavelet Analysis Applying Cubic B-Splines

Waldemar Rakowski

***Abstract*—Wavelet transform algorithms (Mallat's algorithm, à trous algorithm) require input data in the form of a sequence of numbers equal to the signal projection coefficients on a space spanned by integer-translated copies of a scaling function. After sampling of the continuous-time signal, it is most frequently possible to compute only approximated values of the signal projection coefficients by choosing a specific signal approximation. Calculation of the signal projection coefficients based on the signal interpolation by means of cubic B-splines is proposed in the paper.**

***Keywords*—wavelet, scaling function, wavelet analysis, wavelet transform, cubic box spline, digital filters, direct cubic B-spline filter, prefiltering**

I. Introduction

The wavelet analysis is taken advantage of in a significant number of signal processing tasks, e.g. in signal compression, signal singularity detection, denoising, pattern recognition and fractal analysis. The processed signal is most often given as a sequence of samples in the discrete form. However, discrete wavelet transform algorithms (Mallat's algorithm, à trous algorithm) require the input in the form of coefficients of the signal projection onto a space generated by the integer-translated copies of a scaling function rather than samples of the signal. Computing the coefficients of the signal projection on the basis of the samples of the signal is called prefiltering. It is most frequently possible to compute only approximated values of the signal projection coefficients by choosing a specific signal approximation method. Prefiltration based on the signal interpolation by means of cubic B-splines is proposed in the paper. Basic concepts of wavelet analysis are discussed before the scrutiny of the proposed manner of prefiltration.

II. Introduction to Wavelet Analysis

In a wavelet analysis, there are two functions: a scaling function $\phi(t)$ and a wavelet $\psi(t)$ and two digital filters: a lowpass filter h and a highpass filter g. In terms of signal processing, the scaling function is an impulse response of a lowpass filter and the wavelet is an impulse response of a bandpass filter. Both wavelet and scaling function are characterized by fast decay, e.g. exponential, or compact support. Relations between the scaling function, the wavelet and the filters are described by the following two equations:

$$\frac{1}{\sqrt{2}}\ \phi(t/2) = \sum_n\ h[n]\ \phi(t-n), \tag{1}$$

$$\frac{1}{\sqrt{2}}\ \psi(t/2) = \sum_n\ g[n]\ \phi(t-n). \tag{2}$$

The equation (1) is called the dilation equation. A characteristic feature of the scaling function is a non-zero value of the following integral

$$\int_{-\infty}^{+\infty} \phi(t)\ dt \neq 0. \tag{3}$$

It is assumed that $||\phi(t)|| = 1$. Coefficient $1/\sqrt{2}$ ensures preservation of the norm of the scaled function in the space $L^2(R)$. In the frequency domain the scaling equation takes the form of:

$$\Phi(j2\omega) = \frac{1}{\sqrt{2}}\ H(e^{j\omega})\ \Phi(j\omega). \tag{4}$$

The Fourier transform of the scaling function as a transfer function of a lowpass filter is not equal to zero for $\omega = 0$, i.e. $\Phi(j0) \neq 0$. It is assumed that $H(e^{j0}) = \sqrt{2}$ which means that $\sum_{n=-\infty}^{+\infty} h[n] = \sqrt{2}$. The dilation equation allows to calculate the lowpass filter h for a given scaling function $\phi(t)$.

The equation (2) is called the wavelet equation. The main feature of the wavelet is a vanishing integral of the wavelet, i.e.:

$$\int_{-\infty}^{+\infty} \psi(t)\ dt = 0. \tag{5}$$

In the frequency domain, the wavelet equation takes the form of:

$$\Psi(j2\omega) = \frac{1}{\sqrt{2}}\ G(e^{j\omega})\ \Phi(j\omega). \tag{6}$$

Because $\Phi(j0) \neq 0$ then the number of zeros p of the transfer function $G(j\omega)$ for $\omega = 0$ determines the number of zeros of the Fourier transform of $\Psi(j\omega)$ for $\omega = 0$. It is equivalent to the fact that the wavelet has p vanishing moments, thus

$$\int_{-\infty}^{+\infty} t^k f(t)\ dt = 0, \qquad 0 \leq k \leq p-1. \tag{7}$$

This property indicates that the wavelet is orthogonal to the polynomial of degree $k \leq p-1$. The wavelet equation allows to calculate the wavelet for a given scaling function $\phi(t)$ and a given highpass filter g.

A wavelet transform of a continuous-time signal (a continuous spatial variable) is a function of two variables and is a measure of the similarity between the signal $f(t)$ and the wavelet $\psi_{u,s}(t)$ at scale s, shifted to the time instant $t = u$. The formal definition of a wavelet transform of a continuous-time signal is:

$$Wf(u,s) = \int_{-\infty}^{+\infty} f(t)\ \frac{1}{\sqrt{s}}\ \psi\left(\frac{t-u}{s}\right)\ dt \tag{8}$$

$$= \int_{-\infty}^{+\infty} f(t)\ \frac{1}{\sqrt{s}}\ \overline{\psi}\left(\frac{u-t}{s}\right)\ dt, \tag{9}$$

This work was supported by Bialystok University of Technology under grant WZ/1/2014.

W. Rakowski is with the Faculty of Management of Bialystok University of Technology, Poland (e-mail: w.rakowski@pb.edu.pl).

and has the following symbolic notation:

$$Wf(u,s) = f \star \overline{\psi}_s(u). \quad (10)$$

Coefficient $1/\sqrt{s}$ that appears before the scaled and shifted wavelet preserves the norm of the scaled wavelet in the space $L^2(R)$.

A. Dyadic Signal Representation

The signal representation in the form of a continuous-time wavelet transform which is a function of two variables (time and scale), is a highly redundant representation. It is proven [1] that the signal can be reconstructed from a dyadic wavelet transform, in which the scale is not a continuous variable but it is a countable set of dyadic values: $\{s_k\}_{k\in Z}$ where $s_k = 2^k$.

The dyadic wavelet transform is defined as follows:

$$W_{2^k} f(u) = \int_{-\infty}^{+\infty} f(t)\ \frac{1}{\sqrt{2^k}}\ \psi\left(\frac{t-u}{2^k}\right)\ dt \quad (11)$$

$$= f \star \overline{\psi}_{2^k}(u), \quad (12)$$

where $k \in Z$. This transform is a countable set of functions of one variable (time or space).

In the frequency domain the dyadic wavelet transform is expressed as:

$$W_{2^k} F(j\omega) = \sqrt{2^k}\ F(j\omega)\ \overline{\Psi}(j2^k\omega). \quad (13)$$

To calculate the wavelet transform in points $(n,\ 2^k)$, $n,k \in Z$ (linearly spaced points in time and a dyadic scale) a fast algorithm can be used which is called in French ***Algorithme à Trous*** [2] [3].

The first step is to define a rectangle $[t_{min},\ t_{max}] \times [s_{min},\ s_{max}]$ in which the values of the wavelet transform will be calculated. Two rectangles are commonly used: $[0,\ N-1]\times[2,\ s_{max}]$ or $[0,\ 1) \times [s_{min},\ 1]$, where $N = 2^K$. In the case of the rectangle $[0,\ N-1]\times[2,\ s_{max}]$ the wavelet transform values $W_{2^k} f(n)$ are calculated in points $n = 0,1,\ldots,N-1$, for $k = 1,2,\ldots,K$, where $K = \log_2 N$. In the case of the rectangle $[0,\ 1)\times[s_{min},\ 1]$ the wavelet transform values $W_{2^k} f(n/N)$ are calculated in points , $n = 0,1,\ldots,N-1$, for $k = -(K-1), -(K-2),\ldots,0$.

The application of the à trous algorithm to compute the wavelet transform requires to know the digital representation of the signal $f(t)$ in the finest scale, i.e. scale $s = 1$ in the case of window $[0,\ N-1]\times[2,\ s_{max}]$ or scale $s = 1/N$ in the case of window $[0,\ 1)\times[s_{min},\ 1]$. From now the descriptions of the algorithms will concern the rectangle $[0,\ N-1]\times[2,\ s_{max}]$ on plane $0ts$.

B. Discrete Dyadic Wavelet Transform

Let:

$$d_k[n] = W_{2^k} f(n), \quad (14)$$

$$a_K[n] = \int_{-\infty}^{+\infty} f(u)\ \frac{1}{\sqrt{2^K}}\ \phi\left(\frac{u-n}{2^K}\right)\ du \quad (15)$$

$$= f \star \overline{\phi}_{2^K}(n), \quad (16)$$

$k = 1,2,\ldots,K;\ n = 0,1,\ldots,N-1$.

The arrangement of numbers

$$\{\ \{\ d_k[n]\ \}_{\ n=0,1,\ldots,N-1}\ \}_{\ k=1,2,\ldots,K}\ , \quad (17)$$

$$\{\ a_K[n]\ \}_{\ n=0,1,\ldots,N-1}\ , \quad (18)$$

is called the discrete dyadic K-level wavelet transform of signal $f(t)$.

The wavelet coefficient $d_k[n]$ is a measure of the similarity (in the sense of the inner product) of signal $f(t)$ in the surrounding of time instant (a point in space) $t = n$ to the wavelet $\psi\left(\frac{t}{2^k}\right)$ or the value of the signal filtered with a bandpass filter with the impulse response $\overline{\psi}\left(\frac{t}{2^k}\right)$ at $t = n$.

Coefficient $a_K[n]$ is a measure of the similarity of signal $f(t)$ in the surrounding of the time instant $t = n$ to the scaling function $\phi\left(\frac{t}{2^K}\right)$ or the value of the signal filtered by a lowpass filter with the impulse response $\overline{\phi}\left(\frac{t}{2^K}\right)$ at $t = n$.

A fast algorithm for computing a discrete dyadic wavelet transform by using digital filtering is described in section II-D.

The following section describes how to obtain a discrete-time signal $\{a_0[n]\}_{n\in Z}$ which is a digital representation of the signal $f(t)$ at the finest scale and that forms input to the à trous algorithm for the calculation of a discrete dyadic wavelet transform.

C. Digital Representation of the Signal at the Finest Scale

The sequence of numbers equal to the inner product of signal $f(t)$ and the translation of the scaling function $\phi(t-n)$

$$a_0[n] = \int_{-\infty}^{+\infty} f(t)\ \phi(t-n)\ dt, \qquad n \in Z \quad (19)$$

is a discrete representation of signal $f(t)$ at scale 2^0. By introducing in the above equation a new variable $u = t - n$ we obtain

$$a_0[n] = \int_{-\infty}^{+\infty} f(n+u)\ \phi(u)\ du, \qquad n \in Z, \quad (20)$$

showing that the n-th sample of the discrete signal is produced by the averaging signal $f(t)$ in the surrounding of the time instant $t = n$, with the scaling function $\phi(u)$ as a weighting function. Sequence $(a_0[n])$ can also be considered as samples of the signal filtered by the filter with the impulse response equal $\bar{\phi}(t) = \phi(-t)$. Indeed, let the signal $f_0(t)$ be the result of filtration, thus

$$f_0(t) = \int_{-\infty}^{+\infty} f(u)\ \bar{\phi}(t-u)\ du; \quad (21)$$

then $a_0[n] = f_0(n)$. The Fourier transform of signal $f_0(t)$ is given by

$$F_0(j\omega) = F(j\omega)\ \overline{\Phi}(j\omega). \quad (22)$$

D. Algorithme à Trous

In order to describe the algorithm for computing the discrete dyadic wavelet transform, numerical sequences are introduced and defined as follows:

$$a_k[n] = \int_{-\infty}^{+\infty} f(u)\ \frac{1}{\sqrt{2^k}}\ \phi\left(\frac{u-n}{2^k}\right)\ du \quad (23)$$

$$= f \star \overline{\phi}_{2^k}(n), \quad (24)$$

$$d_k[n] = \int_{-\infty}^{+\infty} f(u) \frac{1}{\sqrt{2^k}} \psi\left(\frac{u-n}{2^k}\right) du \quad (25)$$
$$= f \star \overline{\psi}_{2^k}(n), \quad (26)$$

$n = 0, 1, \ldots, N-1;\ k = 1, \ldots, K.$

Let $h_k = (h_k[n])$, $k \geq 0$, be a digital filter with the impulse response created from the impulse response of filter $h = (h[n])$ in such a way that between each two adjacent coefficients of filter h, $2^k - 1$ zeros are inserted, and $\overline{h}_k[n] = h_k[-n]$; analogically, this is applied to filters g_k. The coefficients of the discrete dyadic wavelet transform of signal a_0 can be calculated in the following iterative process:

For $k \geq 0$

$$d_{k+1}[n] = a_k \star \overline{g}_k[n], \quad (27)$$

$$a_{k+1}[n] = a_k \star \overline{h}_k[n]. \quad (28)$$

The proof of this theorem can be found in the book [1]. Figure 1 shows a bank of digital filters implementing one step of the à trous algorithm.

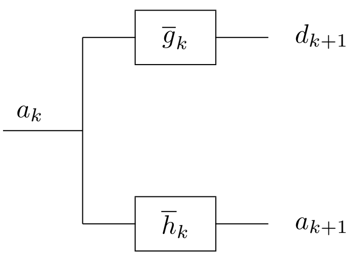

Fig. 1. Filter bank performing one step of the à trous algorithm

The transfer functions of the filters $\overline{g}_k$ and $\overline{h}_k$ are respectively equal to:

$$\overline{G}_k(e^{j\omega}) = \overline{G}(e^{j2^k\omega}), \quad (29)$$

$$\overline{H}_k(e^{j\omega}) = \overline{H}(e^{j2^k\omega}). \quad (30)$$

E. Mallat's Algorithm

Wavelet signal decomposition in accordance with Mallat's algorithm (discrete wavelet transform algorithm) for $k \geq 0$ is described by the computational process below [1]

$$d_{k+1}[n] = a_k \star \overline{g}[2n], \quad (31)$$

$$a_{k+1}[n] = a_k \star \overline{h}[2n]. \quad (32)$$

In contrast to algorithm a trous, in Mallat's algorithm time (space variable) discretization is proportional to the scale equal to 2^k which is expressed by the coefficient 2 next to n in the above equations. Figure 2 shows a bank of digital filters implementing one step of the discrete wavelet transform (Mallat's algorithm).

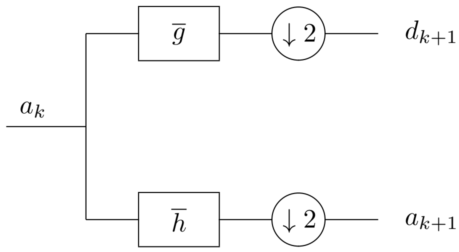

Fig. 2. Filter bank performing one step of the Mallat's algorithm

III. Box Splines

A short and compact support is a desired property of wavelets. A digital filter associated with such a wavelet has a finite and short impulse response which is computationally efficient. The creation of such wavelets is possible by means of functions described with polynomials, referred to as splines, especially the cubic B-spline function. It is described in the following section.

The starting point for the construction of a box spline is defining the gate function $b(t)$ [4]:

$$b(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

The Fourier transform of the gate function is:

$$B(j\omega) = \frac{1 - e^{-j\omega}}{j\omega} \quad (33)$$
$$= e^{-j\omega/2} \frac{\sin(\omega/2)}{\omega/2}. \quad (34)$$

The gate function $b(t)$ is a box spline of zero degree $b_0(t)$. The box spline of m degree is defined recursively as a convolution of the gate function $b(t)$ with the box spline of $m-1$ degree, thus $b_m(t) = b_{m-1} \star b(t)$, $m = 1, 2, \ldots$. The support of the box spline of m degree is included in interval $[0,\ m+1]$.

The convolution in the time domain corresponds to multiplication in the frequency domain of Fourier transforms, so the Fourier transform of the box spline of $m - th$ degree is

$$B_m(j\omega) = e^{-j(m+1)\omega/2} \left[\frac{\sin(\omega/2)}{\omega/2}\right]^{m+1}. \quad (35)$$

The box spline of degree m can be calculated by using the box splines of degree $m-1$

$$b_m(t) = [\ t\ b_{m-1}(t) + (m+1-t)\ b_{m-1}(t-1)\ ]\,/m. \quad (36)$$

The derivative of box spline of degree m can be calculated by using the box spline of degree $m-1$.

$$b'_m(t) = b_{m-1}(t) - b_{m-1}(t-1). \quad (37)$$

The integer translations of box splines $b_m(t-n)$, $n \in Z$ are not orthogonal to one another, thus

$$\int_{-\infty}^{+\infty} b_m(t-n)\ b_m(t-k)\ dt\ \neq\ \delta[n-k]. \quad (38)$$

There is a procedure of orthogonalization, however, splines received as a result do not have a compact support but decay quickly (exponentially).

A. The Box Spline of the Third Degree

The box spline of degree 3 $b_3(t)$ has a support equal to interval $[0,\ 4]$. On each of the subintervals $[k,\ k+1]$, $0 \leq k \leq 3$ the value of the function is determined by another polynomial of the third degree, but a connection of different polynomials in the internal nodes, i.e. at points $t = 1, 2, 3$ is continuous and twice differentiable.

Table I shows polynomials defining the box spline of the third degree, and figure 3 shows the shape of the function.

TABLE I
POLYNOMIALS DESCRIBING THE BOX SPLINE OF THE THIRD DEGREE $b_3(t)$

Interval	$b_3(t)$
$t < 0$	0
$0 \leq t < 1$	$\frac{1}{6}t^3$
$1 \leq t < 2$	$-\frac{1}{2}t^3 + 2t^2 - 2t + \frac{2}{3}$
$2 \leq t < 3$	$-\frac{1}{2}(4-t)^3 + 2(4-t)^2 - 2(4-t) + \frac{2}{3}$
$3 \leq t < 4$	$\frac{1}{6}(4-t)^3$
$t \geq 4$	0

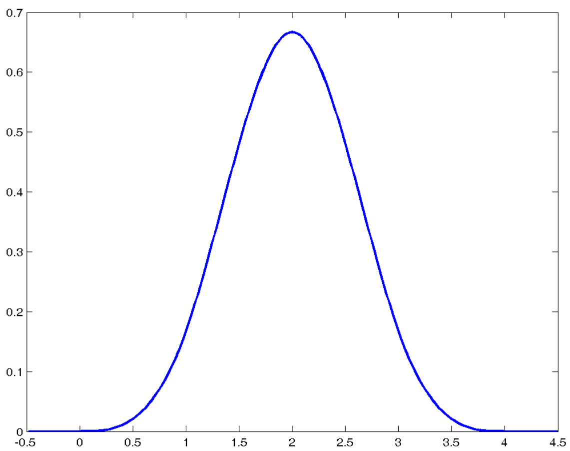

Fig. 3. Box spline of the third degree $b_3(t)$

The Fourier transform of the cubic box spline is:

$$B_3(j\omega) = e^{-j2\omega} \left[\frac{\sin(\omega/2)}{\omega/2} \right]^4. \tag{39}$$

IV. Scaling Function and Wavelet in the Form of Cubic Box Spline

The scaling function is expressed as a box spline function of the third degree shifted to the left by two units, thus:

$$\phi(t) = b_3(t+2). \tag{40}$$

Figure 4 shows the graph of the scaling function and Table II shows the polynomials describing that function.

The Fourier transform of the scaling function is equal to:

$$\Phi(j\omega) = \left[\frac{\sin(\omega/2)}{\omega/2} \right]^4. \tag{41}$$

Figure 5 shows a modulus of the Fourier transform of the scaling function in the form of the cubic box spline. Transfer function of filter h corresponding to the scaling function $\phi(t)$ can be calculated from the dilation equation expressed in the frequency domain (1)

$$H\left(e^{j\omega}\right) = \sqrt{2}\, \frac{\Phi(j2\omega)}{\Phi(j\omega)}. \tag{42}$$

The transfer function of filter h corresponding to the scaling function as a cubic box spline is:

$$H\left(e^{j\omega}\right) = \sqrt{2} \left(\frac{\sin\omega}{\omega} \right)^4 \left[\frac{\sin(\omega/2)}{\omega/2} \right]^{-4} \tag{43}$$

$$= \sqrt{2} \left(\cos\frac{\omega}{2} \right)^4 \tag{44}$$

$$= \frac{\sqrt{2}}{16} \left(e^{j2\omega} + 4\, e^{j\omega} + 6 + 4\, e^{-j\omega} + e^{-j2\omega} \right).$$

The magnitude of transfer function of filter h is shown in Figure 6. The coefficients of filter h are summarized in Table III.

Wavelet $\psi(t)$ can be constructed based on equation (2) by first selecting the highpass filter g. The shortest highpass filter

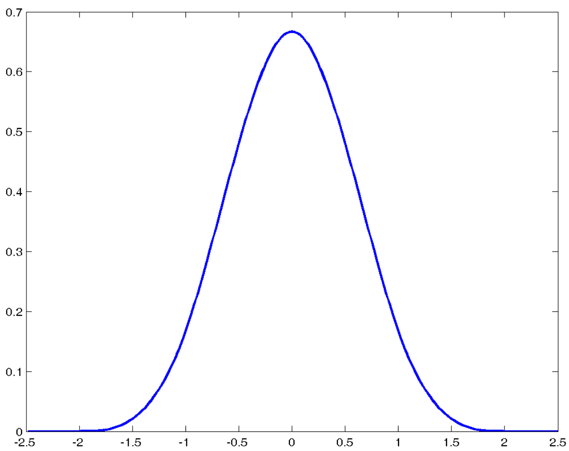

Fig. 4. Scaling function $\phi(t)$ in the form of translation of the cubic box spline

TABLE II
POLYNOMIALS DESCRIBING THE SCALING FUNCTION $\phi(t)$ IN THE FORM OF THE CUBIC BOX SPLINE

Interval	$\phi(t)$
$t < -2$	0
$-2 \leq t < -1$	$(t+2)^3/6$
$-1 \leq t < 0$	$-t^3/2 - t^2 + 2/3$
$0 \leq t < 1$	$t^3/2 - t^2 + 2/3$
$1 \leq t < 2$	$(2-t)^3/6$
$t \geq 2$	0

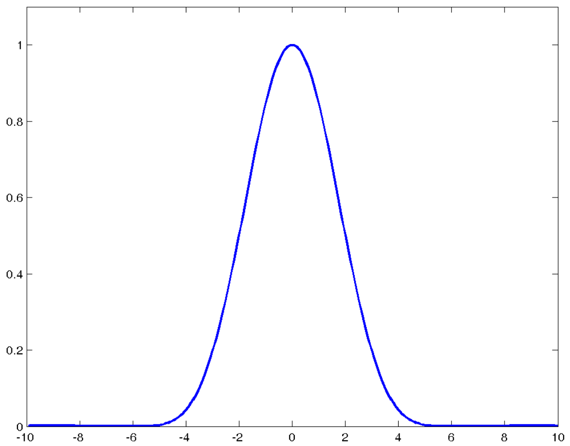

Fig. 5. The magnitude of the Fourier transform of scaling function $\phi(t)$ in the form of cubic box spline

is the Haar filter for which the impulse response coefficients are given in Table IV, and the transfer function is equal to:

$$G\left(e^{j\omega}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e^{-j\omega} \quad (45)$$

$$= -j\sqrt{2}\, e^{-j\frac{\omega}{2}} \sin\frac{\omega}{2}. \quad (46)$$

The wavelet corresponding to this filter is equal to:

$$\psi(t) = -\phi(2t) + \phi(2t-1). \quad (47)$$

From (40) we obtain $\phi(2t) = b_3(2t+2)$ and $\phi(2t-1) = b_3(2t+1)$ so

$$\psi(t) = -b_3(2t+2) + b_3(2t+1) \quad (48)$$

It is easy to see that the support of the wavelet is interval $[-1,\ 1.5]$.

By introducing a new variable $\tau = 2t+2$ equation (48) takes the form of:

$$-\psi\left(\frac{\tau-2}{2}\right) = b_3(\tau) - b_3(\tau-1). \quad (49)$$

The right side of the above equation, according to (37) is the derivative of $b_4(\tau)$, so

$$\psi\left(\frac{\tau-2}{2}\right) = -\frac{d}{d\tau}\, b_4(\tau). \quad (50)$$

The Fourier transform of the wavelet is calculated using the wavelet equation (2)

$$\Psi(j\omega) = \frac{1}{\sqrt{2}}\, G(e^{j\frac{\omega}{2}})\ \Phi(j\omega/2) \quad (51)$$

$$= \frac{-j\omega}{4} e^{-j\frac{\omega}{4}} \left[\frac{\sin(\omega/4)}{\omega/4}\right]^5. \quad (52)$$

The transfer function of filter g expressed by formula (46) has one zero for $\omega = 0$, so the corresponding wavelet has one vanishing moment.

An alternative wavelet $\psi(t)$ can be constructed by selecting the highpass filter g with three impulse response coefficients given in table VI and the transfer function equal:

$$G\left(e^{j\omega}\right) = \frac{\sqrt{2}}{4}e^{j\omega} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}e^{-j\omega} \quad (53)$$

$$= -\sqrt{2}\left(\sin\frac{\omega}{2}\right)^2. \quad (54)$$

The wavelet corresponding to this filter is equal:

$$\psi(t) = \frac{1}{2}\phi(2t+1) - \phi(2t) + \frac{1}{2}\phi(2t-1) \quad (55)$$

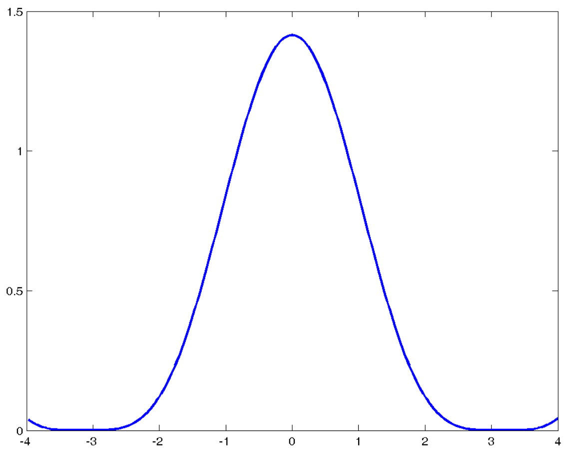

Fig. 6. The magnitude of transfer function of filter h corresponding to the scaling function as a cubic box spline

TABLE III
THE COEFFICIENTS OF THE LOWPASS FILTER $h[n]$ CORRESPONDING TO THE SCALING FUNCTION $\phi(t)$ AS A CUBIC BOX SPLINE

n	-2	-1	0	1	2
$h[n]$	$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{4}$	$\frac{3\sqrt{2}}{8}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{16}$

TABLE IV
COEFFICIENTS OF THE HIGHPASS FILTER $g[n]$ CORRESPONDING TO WAVELET $\psi(t)$ IN THE FORM OF CUBIC SPLINE (47)

n	0	1
$g[n]$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

TABLE V
THE POLYNOMIALS DESCRIBING WAVELET $\psi(t)$ IN THE FORM OF A CUBIC SPLINE CORRESPONDING TO FILTER g WITH COEFFICIENTS GIVEN IN TABLE IV

Interval	$\psi(t)$
$t < -1$	0
$-1 \le t < -0.5$	$-\frac{4}{3}(1+t)^3$
$-0.5 \le t < 0$	$\frac{16}{3}t^3 + 6t^2 + t - \frac{1}{2}$
$0 \le t < 0.5$	$-8t^3 + 6t^2 + t - \frac{1}{2}$
$0.5 \le t < 1$	$\frac{16}{3}t^3 - 14t^2 + 11t - \frac{13}{6}$
$1 \le t < 1.5$	$\frac{4}{3}(\frac{3}{2} - t)^3$
$t \ge 1.5$	0

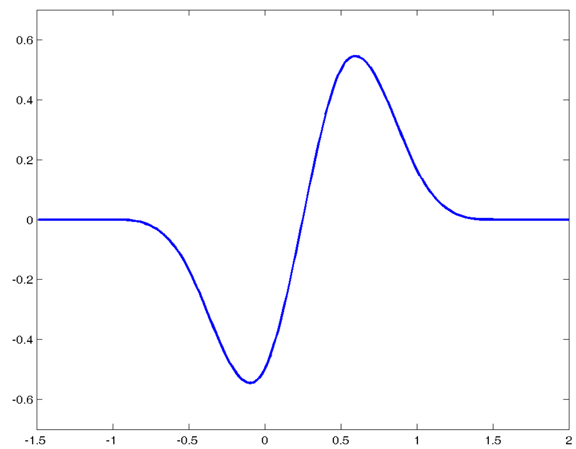

Fig. 7. The wavelet $\psi(t)$ in the form of a cubic spline corresponding to filter g with coefficients given in Table IV

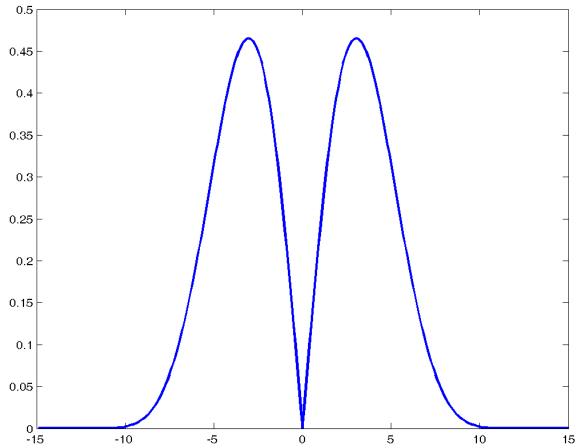

Fig. 8. Magnitude of the Fourier transform of the wavelet $\psi(t)$ in the form of cubic spline corresponding to the filter g with coefficients given in Table IV

and has support $[-1.5,\ 1.5]$. Filter g with the transfer function (54) has two zeros for $\omega = 0$, so the corresponding wavelet has two vanishing moments. In Figure 9 it can be seen that the wavelet has three local extremes. The Fourier transform of the wavelet is equal

$$\Psi(j\omega) = \frac{(j\omega)^2}{4} \left[\frac{\sin(\omega/4)}{\omega/4}\right]^6 . \qquad (56)$$

TABLE VI
THE COEFFICIENTS OF THE HIGHPASS FILTER $g[n]$ CORRESPONDING TO WAVELET $\psi(t)$ IN THE FORM OF A SPLINE FUNCTION OF THE THIRD DEGREE (55)

n	-1	0	1
$g[n]$	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{4}$

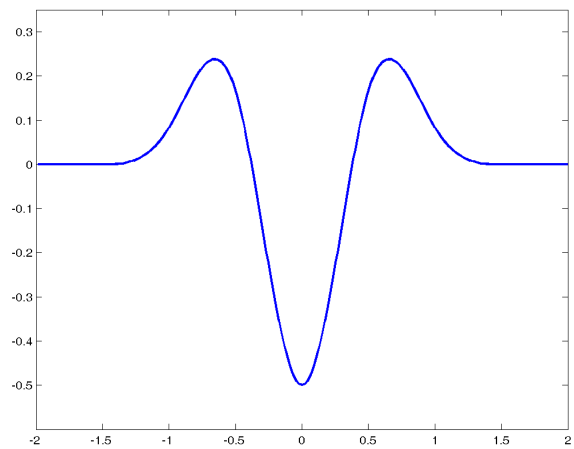

Fig. 9. Wavelet $\psi(t)$ in the form of a cubic spline corresponding to filter g with coefficients given in Table VI

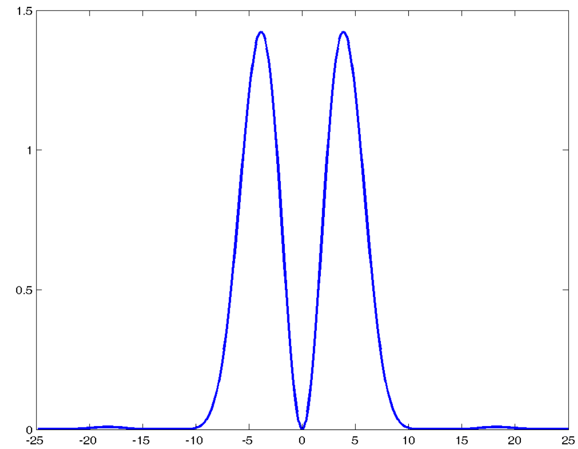

Fig. 10. The magnitude of Fourier transform of wavelet $\psi(t)$ in the form of a cubic spline corresponding to filter g with coefficients given in Table VI

Expression

$$\frac{1}{2} \left[\frac{\sin(\omega/4)}{\omega/4}\right]^6 \qquad (57)$$

is, in accordance with (35) the Fourier transform of $b_5\left(2(t+\frac{3}{2})\right)$. Value $(j\omega)^2$ in Fourier transform corresponds to the second derivative of function $b_5\left(2(t+\frac{3}{2})\right)$. Thus, the wavelet is equal to:

$$\psi(t) = \frac{d^2}{dt^2}\left\{\frac{1}{2}\ b_5\left[2\left(t+\frac{3}{2}\right)\right]\right\}. \qquad (58)$$

V. PREFILTERING

Wavelet transform algorithms require input data in the form of a sequence of coefficients described with the following formula:

$$a_0[n] = \int_{-\infty}^{+\infty} f(t)\ \phi(t-n)\ dt, \qquad n \in Z. \qquad (59)$$

The signal $f(t)$ is most frequently given in the form of the samples $f[n] = f(n)$. Therefore, to compute the right side of the above equation it is necessary to choose a specific approximation of the signal $f(t)$.

Assuming that:

- the signal f(t) has a limited energy, i.e.:

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, dt < +\infty, \qquad (60)$$

- the support of the Fourier transform is contained in the finite interval $[-\omega_m, \ \omega_m]$, i.e.:

$$\left| \int_{-\infty}^{+\infty} f(t) \, e^{-j\omega t} \, dt \right| = 0 \quad dla \ \ |\omega| > \omega_m. \quad (61)$$

- the sampling rate is equal to T

$$T < \frac{\pi}{\omega_m} \qquad (62)$$

then according with the sampling theorem [5] [6] the signal can be described with the following formula:

$$f(t) = \sum_{n=-\infty}^{+\infty} f(nT) \, \mathrm{sinc}\Big(\frac{t - nT}{T}\Big), \qquad (63)$$

where $\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$.

In order to simplify the formula it is assumed that the independent variable t has been scaled in such a way that the condition (62) is fulfilled with the sampling rate $T = 1$. Then the equation (63) takes the form:

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n) \, \mathrm{sinc}(t - n), \qquad (64)$$

and the signal band is contained in the interval $[-\pi, \ \pi]$.

The function $\mathrm{sinc}(t)$ belongs to the so-called interpolating functions because $\mathrm{sinc}(n) = \delta[n]$, $n \in Z$, where

$$\delta[n] = \begin{cases} 1 & \text{jeli } n = 0 \\ 0 & \text{jeli } n \neq 0. \end{cases}$$

The function $\mathrm{sinc}(t)$ called cardinal sine shifted by integer numbers create an orthogonal basis of the space containing functions with the signal band included in the interval $[-\pi, \ \pi]$. However, wavelets created on the basis of that function are not in reality taken advantage of. The reason for this is the fact that the function $\mathrm{sinc}(t)$ disappears slowly and the digital filters associated with the function and its linear combinations are filters with infinite and slowly decaying impulse response. Figure 11 shows the function cardinal sine at the forefront of the function called *cardinal cubic spline* [7]. The function cardinal cubic spline, represented further as $\eta(t)$, belongs to the space of the cubic splines of the third degree. In each of the intervals $[n, \ n+1)$, $n \in Z$ the function is characterized by a different polynomial of the third degree but the connections in points $t = n$, $n \in Z$ are twice differentiable. The function, similarly to $\mathrm{sinc}(t)$ is an interpolating function, i.e. $\eta(n) = \delta[n]$, $n \in Z$ and her shifts by integer numbers $\{\eta(t-n)\}_{n \in Z}$ create a basis of splines of the third degree. It means that if the function $f(t)$ belongs to such a space then:

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n) \, \eta(t - n). \qquad (65)$$

The function cardinal cubic spline, similarly to the function $\mathrm{sinc}(t)$ has an unlimited support. However, it disappears fast in an exponential way. The digital filter associated with the function has an infinite impulse response.

In order to simplify formulas we define a cubic B-spline symmetric in zero:

$$\beta(t) = b_3(t + 2). \qquad (66)$$

Computationally it is more useful to describe the spline of the third degree as a linear combination of the (symmetric) cubic B-spline shifted by integer numbers according to the formula:

$$f(t) = \sum_{n=-\infty}^{+\infty} c[n] \, \beta(t - n). \qquad (67)$$

The following section describes the way of computing the coefficients $c[n]$, $n \in Z$ in the equation (67) on the basis of samples of the signal $f[n]$, $n \in Z$ called direct cubic B-Spline transform.

A. Direct Cubic B-spline Transform

Direct cubic B-spline transform [7] [8] serves the purpose of computing the coefficients $c[n]$, $n \in Z$ in the signal expansion in the cubic B-spline basis (formula (67)). The starting point for the creation of the filter performing the direct cubic B-spline transform is describing of the cubic B-spline-interpolation function (cardinal cubic spline) in the form of a linear combination of cubic B-spline functions shifted by integer numbers.

$$\eta(t) = \sum_{k=-\infty}^{\infty} p[k] \, \beta(t - n). \qquad (68)$$

In points $t = n$, $n \in Z$ we obtain

$$\eta(n) = \sum_{k=-\infty}^{\infty} p[k] \, \beta(n - k). \qquad (69)$$

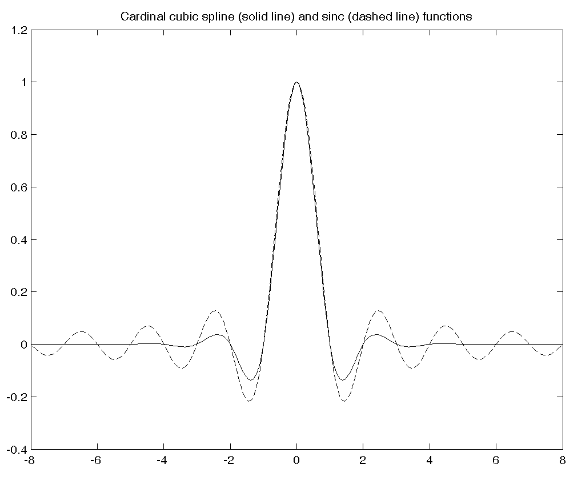

Fig. 11. Cardinal sine (sinc) and cardinal cubic spline functions

After defining the sequence of numbers

$$b[n] \ = \ \beta(n), \quad n \in Z \tag{70}$$

and taking advantage of the fact that $\eta(n) = \delta[n]$ we obtain the equation

$$\begin{aligned} \delta[n] \ &= \ \sum_{k=-\infty}^{\infty} p[k] \ b[n-k] && (71)\\ &= \ p * b[n]. && (72) \end{aligned}$$

It follows from the equation (72) that the sequence of numbers $\{p[k]\}_{k \in Z}$ is a reverse filter to filter b. The filter b called *indirect cubic B-spline filter* is a filter with a finite impulse response specified in table VII. The filter p called *direct cubic B-spline filter* is a filter with an infinite impulse response as it is reverse to a filter with a finite response. The manner of computing coefficients for this filter and the ways of filtration by means of this filter are described in the following section.

TABLE VII
IMPULSE RESPONSE COEFFICIENTS OF FILTER b

n	-1	0	1
$b[n]$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$

When we assume that we know the coefficients of the impulse response of the filter p we use the equations below to obtain the coefficients of the expansion of the signal $f(t)$ in the basis of cubic B-spline translations.

$$\begin{aligned} f(t) \ &= \ \sum_{k=-\infty}^{\infty} f[k] \ \eta(t-k) && (73)\\ &= \ \sum_{k=-\infty}^{\infty} f[k] \sum_{m=-\infty}^{\infty} p[m] \ \beta(t-m-k) && (74)\\ &= \ \sum_{m=-\infty}^{\infty} \ \sum_{k=-\infty}^{\infty} f[k] \ p[m] \ \beta(t-m-k) && (75)\\ &= \ \sum_{n=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} f[k] \ p[n-k] \Big) \ \beta(t-n) && (76)\\ &= \ \sum_{n=-\infty}^{\infty} (f * p[n]) \ \beta(t-n). && (77) \end{aligned}$$

The above equations show that the function $f(t)$ in the basis $\{\beta(t-n)\}_{n \in Z}$ is described by the formula

$$f(t) \ = \ \sum_{n=-\infty}^{\infty} c[n] \ \beta(t-n), \tag{78}$$

in which

$$c[n] \ = \ f * p[n], \quad n \in Z, \tag{79}$$

which means that the coefficients of the expansion of the signal $f(t)$ are a convolution of the samples of the signal $\{f[n]\}_{n \in Z}$ with the filter p.

B. Direct Cubic B-spline Filter

Z transform of both sides of the equation (72) gives the equation

$$1 \ = \ P(z) \ B(z), \tag{80}$$

which in turn gives the equation

$$P(z) \ = \ \frac{1}{B(z)}. \tag{81}$$

Z transform of the indirect cubic B-spline filter is equal to

$$B(z) \ = \ \frac{1}{6} z + \frac{4}{6} + \frac{1}{6} z^{-1}; \tag{82}$$

Z transform of the direct cubic B-spline filter is equal to

$$\begin{aligned} P(z) \ &= \ \frac{1}{B(z)} && (83)\\ &= \ \frac{6}{z + 4 + z^{-1}} && (84)\\ &= \ \frac{6z}{(z-a)(z-a^{-1})}, && (85) \end{aligned}$$

where $a = \sqrt{3} - 2$. The right side of the equation (85) can be transformed to the form

$$\begin{aligned} P(z) \ &= \ \frac{6}{1-a^2} \Big(\frac{-a^2}{z-a} + \frac{1}{z-a^{-1}} \Big) && (86)\\ &= \ \frac{-6a}{1-a^2} \Big(\frac{1}{1-az^{-1}} + \frac{1}{1-az} - 1 \Big). && (87) \end{aligned}$$

It is easy to confirm that

$$\frac{1}{1-az^{-1}} \ = \ \sum_{n=-\infty}^{\infty} a^n z^{-n}, \quad \text{if } |z| > |a|. \tag{88}$$

It means that in the given region of convergence the above series is the Z transform of the sequence $\{a^n u[n]\}_{n \in Z}$, where

$$u[n] = \begin{cases} 0 & \text{jeli if } n < 0 \\ 1 & \text{jeli if } n \geq 0. \end{cases}$$

The sequence $\left\{\frac{-6a}{1-a^2} a^n u[n]\right\}_{n \in Z}$ is the causal component of the impulse response of the direct B-cubic spline filter shown in the Figure 12.

The second component of the right side of the equation(87) is the Z transform of the sequence $\{a^{-n} u[-n]\}_{n \in Z}$ w in the region $|z| > |a^{-1}|$. The common region of both transforms $|a| < |z| < |a^{-1}|$, where $a = -0.267949$, includes a unit circle and taking into the consideration the third component of the right side of the equation (87) we obtain the impulse response of the direct cubic B-spline filter

$$\begin{aligned} p[n] \ &= \ \frac{-6a}{1-a^2} \Big(a^n u[n] + a^{-n} u[-n] - \delta[n] \Big) && (89)\\ &= \ \frac{-6a}{1-a^2} \ a^{|n|} && (90) \end{aligned}$$

$n \in Z$.

1) Filtering algorithms: We assume that the $f[n]$ has the length N and it is determined for indexes $n = 0,\ 1,\ \dots,\ N-1$. In order to avoid possible sudden drops or increases of the values of the wavelet coefficients at the beginning and at the end of the signal the signal is lengthened twice, made symmetric and periodised. In the case of filters with an odd number of coefficients the symmetrisation shown in Figure 13 is most often performed. The number of samples in the period after the symmetrisation is equal to $2N-2$.

There are 4 ways of the filtration of the signal $f[n]$ by means of the filter $p[n]$. The first manner consists in computing the convolution of sequences $f[n]$ i $\bar{p}[n]$, where $\bar{p}[n] = p[-n]$, $n \in Z$. The impulse response of the filter p is symmetric in zero, i.e. $p[-n] = p[n]$, therefore we should compute the convolution of sequences $f[n]$ and $p[n]$. The sequence $p[n]$ is infinite but it disappears exponentially. Therefore we can take into account a small number of initial coefficients of the sequence. Later we will use the sequnce $p_L[n]$ defined in the following way:

$$p_L[n] = \begin{cases} p[n] & \text{if } |n| < L \\ 0 & \text{if } |n| \geq L. \end{cases}$$

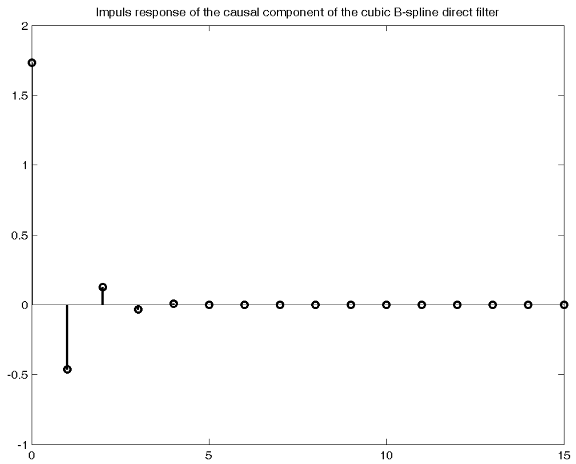

Fig. 12. Causal component of the direct cubic B-spline filter impulse response

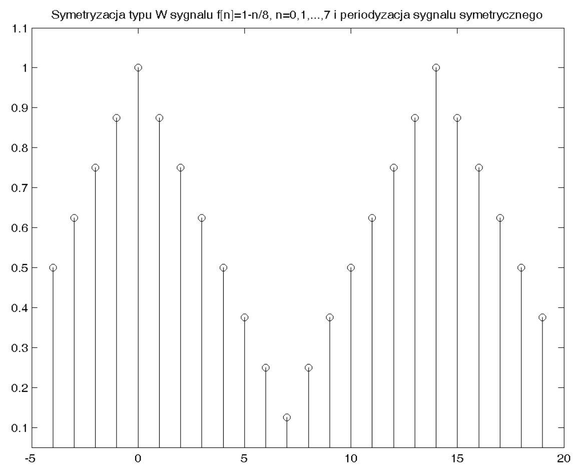

Fig. 13. Symmetrisation and periodisation of the signal $f[n] = 1 - n/8$, $n = 0, 1, \dots, 7$

The second manner of filtration consists in using the algorithm FFT, i.e. computing discrete Fourier transforms of the sequence $f[n]$ and the sequence $p_L[n]$, computing the sequence being the product of the components of the transforms and computing the inverse discrete Fourier transform of such a sequence. Such a manner is not rival in relation to the first manner because the filter p_L is not long.

The third manner consists in recursive filtration on the basis of the difference equations resulting from the Z transform of the filter p. On the basis of the first component of the right hand side of the equation (87) we obtain the following equation in the of the Z transform:

$$C_1(z) = \frac{1}{1 - az^{-1}} F(z), \tag{91}$$

The following difference equation corresponds with the obtained equation:

$$c_1[n] = a\ c_1[n-1] + f[n], \quad n = 1,\ 2,\ \dots,\ 2N-3 \tag{92}$$

The initial condition i.e. the component $c_1[0]$ is computed according to to the formula below for the manner of symmetrization and periodization of the signal $f[n]$ as shown in the Figure 13

$$c_1[0] = \sum_{l=0}^{L-1} f[l]\ p[l] \tag{93}$$

On the basis of the second component of the right side of the equation (87) we obtain the equation:

$$C_2(z) = \frac{1}{1 - az} F(z), \tag{94}$$

The following difference equation corresponds with the obtained equation:

$$c_2[n] = a\ c_2[n+1] + f[n], \quad n = 2N-4,\ 2N-5,\ \dots,\ 0. \tag{95}$$

The initial condition i.e. the component $c_2[2N-3]$ is computed according to to the formula

$$c_2[2N-3] = f[2N-3]\ p[0] + \sum_{l=1}^{L-1} f[l-1]\ p[l]. \tag{96}$$

The third component of the right side of the equation (87) gives the equation

$$C_3(z) = -\ 1 \cdot F(z), \tag{97}$$

which has the solution

$$c_3[n] = -\ \delta * f[n] = -\ f[n], \quad n = 0,\ 1,\ \dots,\ 2N-3. \tag{98}$$

The resulting signal of the filtration

$$c[n] = f * p[n] \tag{99}$$

$$= \frac{-6a}{1-a^2} \left(c_1[n] + c_2[n] - f[n] \right), \tag{100}$$

$n = 0,\ 1,\ \dots,\ 2N-3.$

The forth manner of filtration results from describing the Z tranform of the sequence $c[n]$ in the form

$$C(z) = \frac{6z}{(z-a)(z-a^{-1})}\ F(z) \quad (101)$$

$$= \frac{-a}{1-az}\ C_1(z), \quad (102)$$

where

$$C_1(z) = \frac{1}{1-az^{-1}}\ \big[\ 6\ F(z)\big]. \quad (103)$$

The recursive solution of the equation (103) (the causal component of the filter response p) is

$$c_1[n] = ac_1[n-1] + 6f[n], \quad n = 1,\ 2,\ \ldots,\ 2N-3. \quad (104)$$

Taking into account the response to the non-causal component of the filter p (equation (102)) gives the final solution in the form

$$c[n] = a(c[n{+}1] - c_1[n]), \quad n = 2N-4,\ 2N-5,\ \ldots,\ 0. \quad (105)$$

The initial conditions of $c_1[0]$ i $c[2N-3]$ are the same as in the third manner of filtration described above (element the component $c[2N-3]$ is equal to the component $c_2[2N-3]$). The last manner of filtration is computationally most efficient of all manners described above.

C. Prefiltering for Cubic B-spline Signal Interpolation

In order to compute approximate values of the coefficients (59) the signal $f(t)$ is replaced with the interpolating function of the sample $f[n]$ equal (in accordance with (78)) $\sum_k c[k]\ \beta(t-k)$. Additionally after taking into account the assumption (40) we receive

$$\tilde{a}_0[n] = \int_{-\infty}^{+\infty} \Big(\sum_k c[k]\ \beta(t-k)\Big)\ \beta(t-n)\ dt \quad (106)$$

$$= \sum_k c[k] \int_{-\infty}^{+\infty} \beta(t-k)\ \beta(t-n)\ dt \quad (107)$$

$$= \sum_k c[k] \int_{-\infty}^{+\infty} \beta(\tau)\ \beta(\tau-n+k)\ d\tau \quad (108)$$

$$= \sum_k c[k] \int_{-\infty}^{+\infty} \beta(\tau)\ \beta(n-k-\tau)\ d\tau \quad (109)$$

$$= \sum_k c[k]\ \beta * \beta(n-k). \quad (110)$$

By definition, $\beta(t)$ is cubic B-spline therefore $\beta * \beta(t)$ is B-spline of order 7. Taking for B-spline of order 7 symbol $\beta^7(t)$ and defining the sequence of coefficients

$$b^7[n] = \beta^7(n), \quad n \in Z \quad (111)$$

we receive

$$\tilde{a}_0[n] = \sum_k c[k]\ \beta^7(n-k) \quad (112)$$

$$= \sum_k c[k]\ b^7[n-k] \quad (113)$$

$$= c * b^7[n]. \quad (114)$$

The table VIII shows non-zero coefficients of the filter b^7.

TABLE VIII
THE VALUES OF THE NON-ZERO COEFFICIENTS OF THE FILTER b^7

$\|n\|$	0	1	2	3
$b^7[\|n\|]$	$\frac{2416}{5040}$	$\frac{1191}{5040}$	$\frac{120}{5040}$	$\frac{1}{5040}$

VI. CONCLUSION

Is a prefiltering in wavelet analysis significant for practical applications? Gilbert Strang described the lack of prefiltering as *It is a wavelet crime* [9]. The experiments carried out by the author indicate that the lack of prefiltering does not make it difficult to identify the points where the signal changes its value rapidly. However, prefiltering is significant, for example, in the computation of the measure of the signal singularities in the form of Lipschitz exponent. A practical approach to the topic of prefiltering in wavelet analysis is taken in this article. Prefiltering is described in the language of digital signal processing with practical and effective computational algorithms concerning the applications of wavelets in the form of cubic box splines. In the subject literature one can find advanced mathematical analysis dealing with estimates of errors resulting from the lack of prefiltering and with various manners of prefiltering in the cases of applying wavelets other than cubic box splines (both orthogonal and biorthogonal) [10] [11] [12].

REFERENCES

[1] S. Mallat, *A Wavelet Tour of Signal Processing: The Sparce Way, Third Edition*. Academic Press, 2009.
[2] M. Holschneider, R. Kronland-Martinet, J. Morlet, and P. Tchamitchian, “A real-time algorithm for signal analysis with help of the wavelet transform,” in *Wavelets, Time-Frequency Methods and Phase Space, Springer-Verlag*, 1989.
[3] M. J. Shensa, “The discrete wavelet transform: Wedding the a trous and mallat algorithms,” *IEEE Transactions on Signal Processing, vol. 40, no. 10, pp. 2464-2482*, 1992.
[4] M. Unser, “Splines: A perfect fit for signal and image processing,” *IEEE Signal Processing Magazine, pp. 22-38*, 1999.
[5] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals and Systems*. Prentice-Hall International, Inc., 2/E, 1997.
[6] J. G. Proakis and D. G. Manolakis, *Digital Signal Processing*. Pearson Prentice Hall, 2007.
[7] M. Unser, A. Aldroubi, and M. Eden, “B-spline signal processing: Part i - theory,” *IEEE Transactions on Signal Processing, Vol. 41, No. 2, pp. 821-833*, 1993.
[8] ——, “B-spline signal processing: Part ii - efficient design and applications,” *IEEE Transactions on Signal Processing, Vol. 41, No. 2, pp. 834 - 848*, 1993.
[9] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Wellesley - Cambridge Press, 1996.
[10] B. R. Johnson and J. L. Kinsey, “Quadrature prefilters for discrete wavelet transform,” *IEEE Transactions on Signal Processing, Vol. 48, No.3, pp. 873 - 875*, 2000.
[11] J. Zhank and Z. Bao, “Initialization of orthogonal discrete wavelet transforms,” *IEEE Transactions on Signal Processing, Vol. 48, No. 5, pp. 1474 - 1477*, 2000.
[12] S. Ericsson and N. Grip, “Efficient wavelet prefilters with optimal time-shifts,” *IEEE Transactions on Signal Processing, Vol. 53, No. 7, 2451 - 2461*, 2005.